Section 5.4 Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Example 1. Find the exact value of $\tan 15^o$

$$\tan 15^o = \tan(45^o - 30^o) = \frac{\tan 45^o - \tan 30^o}{1 + \tan 45^o \tan 30^o}$$

$$\tan(45^o) = 1$$
$$\tan(30^o) = \frac{1}{\sqrt{3}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{3}}}$$

$$1 + \frac{1}{\sqrt{3}}$$

$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} \cdot \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$$

$$= \frac{(\sqrt{3})^2 - \sqrt{3} - \sqrt{3} + 1}{(\sqrt{3})^2 - (1)^2} = \frac{3 - 2\sqrt{3} + 1}{3 - 1} = \frac{4 - 2\sqrt{3}}{2}$$
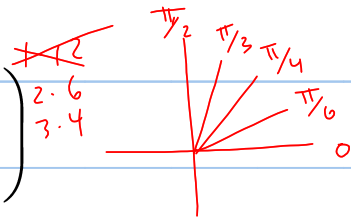
$$= \frac{\cancel{2}(2 - \sqrt{3})}{\cancel{2}}$$

$$= \boxed{2 - \sqrt{3}}$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

Example 2. Find the exact value of $\cos\left(\dfrac{7\pi}{12}\right)$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

Example 3: Find the exact value of $\sin 78^o \cos 18^o - \cos 78^o \sin 18^o$

$\sin 78^o \cos 18^o - \cos 78^o \sin 18^o = \sin(78^\circ - 18^\circ)$

$$= \sin(60^\circ)$$

$$= \frac{\sqrt{3}}{2}$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Example 4: Find the exact value of $\dfrac{\tan\left(\dfrac{7\pi}{5}\right) - \tan\left(\dfrac{23\pi}{20}\right)}{1 + \tan\left(\dfrac{7\pi}{5}\right)\tan\left(\dfrac{23\pi}{20}\right)}$

$$= \tan\left(\frac{7\pi}{5} - \frac{23\pi}{20}\right) = \tan\left(\frac{28\pi}{20} - \frac{23\pi}{20}\right)$$

$$= \tan\left(\frac{5\pi}{20}\right)$$

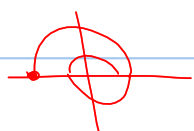$$= \tan\left(\frac{\pi}{4}\right) = 1$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

Example 5. Simplify $\sin(x + 3\pi)$

$$\sin(x + 3\pi) = \sin x \cos 3\pi + \cos x \sin 3\pi$$

$$= (\sin x)(-1) + (\cos x)(0)$$

$$= -\sin x + 0$$

$$= \boxed{-\sin x}$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

LHS   RHS

Example 6. Verify the identity $\sin(90^o - x) = \cos x$

LHS: $\sin(90^\circ - x) = \sin 90^\circ \cos x - \cos 90^\circ \sin x$

$= (1)\cos x - (0)\sin x$

$= \cos x - 0$

$= \cos x = RHS$

$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

Example 7. Solve $\sin\left(x + \dfrac{\pi}{6}\right) - \sin\left(x - \dfrac{\pi}{6}\right) = \dfrac{1}{2}$ on the interval $[0, 2\pi)$

$\Rightarrow \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} - \left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right) = \frac{1}{2}$
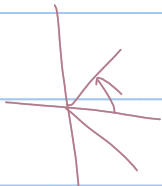
$\Rightarrow \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} - \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} = \frac{1}{2}$

$\Rightarrow \qquad 2 \cos x \sin \frac{\pi}{6} = \frac{1}{2}$

$\left(\frac{1}{2}\right) 2 \cos x = \frac{1}{2}$

$\cos x = \frac{1}{2}$

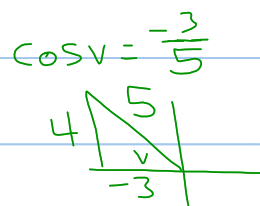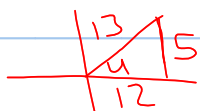$\boxed{x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}}$

Example 8. If $\sin u = \dfrac{5}{13}$ *where* $0 < u < \dfrac{\pi}{2}$ *and* $\cos v = \dfrac{-3}{5}$ *where* $\dfrac{\pi}{2} < v < \pi$

$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

Find the exact value of $\cos(u + v)$

$\sin u = \frac{5}{13}$

$\cos v = \frac{-3}{5}$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$= \left(\frac{12}{13}\right)\left(\frac{-3}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$

$= \frac{-36}{65} - \frac{20}{65} = \boxed{\frac{-56}{65}}$