**54.** **(a)** $f(2) = x^2 - a^2 - x$

$= (2)^2 - a^2 2$

$= 4 - 2a^2$

**(b)** $f(2) = 4 - 2x^2$

$= 4 - 2(2)^2$

$= 4 - 8 = -4$

**(c)** For $x \neq 2$, $f$ is continuous. For $x = 2$, we have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) = -4$ as long as $a = \pm 2$.

**55.** **(a)** $g(x) = \dfrac{x^3}{x^2} = x$

**(b)** $\dfrac{f(x)}{g(x)} = \dfrac{x^3 - 2x^2 + 1}{x^2 + 3} = \dfrac{1}{x}$

$= \dfrac{x^3 - 2x^2 + 1}{x^3 + 3x}$

$\dfrac{x^3}{x^3} = 1$

# Chapter 3
## Derivatives

## Section 3.1 Derivative of a Function (pp. 99–108)

### Exploration 1 Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since $y$ is measured in inches and $x$ is measured in days, the derivative $\dfrac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the $y$-axis in Figure 3.3b.

2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of 5 inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

5. The $y$-coordinate of point $C$ gives the maximum depth of the water level in the ditch over the 7-day period, while the $x$-coordinate of $C$ gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at $C'$, indicating that this is when the water level stops rising and begins falling.

6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

### Quick Review 3.1

1. $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{4} = \lim_{h \to 0} \dfrac{(4 + 4h + h^2) - 4}{h}$

$= \lim_{h \to 0} 4 + h$

$= 4 + 0 = 4$

2. $\lim_{x \to 2^+} \dfrac{x+3}{2} = \dfrac{2+3}{2} = \dfrac{5}{2}$

3. Since $\dfrac{|y|}{y} = -1$ for $y < 0$, $\lim_{y \to 0^-} \dfrac{|y|}{y} = -1$.

4. $\lim_{x \to 4} \dfrac{2x - 8}{\sqrt{x} - 2} = \lim_{x \to 4} \dfrac{2(\sqrt{x} + 2)(\sqrt{x} - 2)}{\sqrt{x} - 2}$

$= \lim_{h \to 4} 2(\sqrt{x} + 2) = 2(\sqrt{4} + 2) = 8$

5. The vertex of the parabola is at (0, 1). The slope of the line through (0, 1) and another point $(h, h^2 + 1)$ on the parabola is $\dfrac{(h^2 + 1) - 1}{h - 0} = h$. Since $\lim_{h \to 0} h = 0$, the slope of the line tangent to the parabola at its vertex is 0.

6. Use the graph of $f$ in the window $[-6, 6]$ by $[-4, 4]$ to find that (0, 2) is the coordinate of the high point and (2, –2) is the coordinate of the low point. Therefore, $f$ is increasing on $(-\infty, 0]$ and $[2, \infty)$.

7. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1)^2 = (1-1)^2 = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1 + 2 = 3$

8. $\lim_{h \to 0^+} f(1+h) = \lim_{x \to 1^+} f(x) = 0$

9. No, the two one-sided limits are different (see Exercise 7).

10. No, $f$ is discontinuous at $x = 1$ because $\lim_{x \to 1} f(x)$ does not exist.

**54.** **(a)** $f(2) = x^2 - a^2 - x$

$= (2)^2 - a^2 2$

$= 4 - 2a^2$

**(b)** $f(2) = 4 - 2x^2$

$= 4 - 2(2)^2$

$= 4 - 8 = -4$

**(c)** For $x \neq 2, f$ is continuous. For $x = 2$, we have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = -4$ as long as $a = \pm 2$.

**55.** **(a)** $g(x) = \frac{x^3}{x^2} = x$

**(b)** $\frac{f(x)}{g(x)} = \frac{x^3 - 2x^2 + 1}{x^2 + 3} = \frac{1}{x}$

$= \frac{x^3 - 2x^2 + 1}{x^3 + 3x}$

$\frac{x^3}{x^3} = 1$

# Chapter 3
## Derivatives

## Section 3.1 Derivative of a Function (pp. 99–108)

### Exploration 1 Reading the Graphs

1. The graph in Figure 3.3b represents the rate of change of the depth of the water in the ditch with respect to time. Since $y$ is measured in inches and $x$ is measured in days, the derivative $\frac{dy}{dx}$ would be measured in inches per day. Those are the units that should be used along the $y$-axis in Figure 3.3b.

2. The water in the ditch is 1 inch deep at the start of the first day and rising rapidly. It continues to rise, at a gradually decreasing rate, until the end of the second day, when it achieves a maximum depth of 5 inches. During days 3, 4, 5, and 6, the water level goes down, until it reaches a depth of 1 inch at the end of day 6. During the seventh day it rises again, almost to a depth of 2 inches.

3. The weather appears to have been wettest at the beginning of day 1 (when the water level was rising fastest) and driest at the end of day 4 (when the water level was declining the fastest).

4. The highest point on the graph of the derivative shows where the water is rising the fastest, while the lowest point (most negative) on the graph of the derivative shows where the water is declining the fastest.

5. The $y$-coordinate of point $C$ gives the maximum depth of the water level in the ditch over the 7-day period, while the $x$-coordinate of $C$ gives the time during the 7-day period that the maximum depth occurred. The derivative of the function changes sign from positive to negative at $C'$, indicating that this is when the water level stops rising and begins falling.

6. Water continues to run down sides of hills and through underground streams long after the rain has stopped falling. Depending on how much high ground is located near the ditch, water from the first day's rain could still be flowing into the ditch several days later. Engineers responsible for flood control of major rivers must take this into consideration when they predict when floodwaters will "crest," and at what levels.

### Quick Review 3.1

**1.** $\lim_{h\to 0} \frac{(2+h)^2 - 4}{4} = \lim_{h\to 0} \frac{(4 + 4h + h^2) - 4}{h}$

$= \lim_{h\to 0} 4 + h$

$= 4 + 0 = 4$

[redacted]

**3.** Since $\frac{|y|}{y} = -1$ for $y < 0$, $\lim_{y\to 0^-} \frac{|y|}{y} = -1$.

[redacted]

$= \lim_{h\to 4} 2(\sqrt{x} + 2) = 2(\sqrt{4} + 2) = 8$

**5.** The vertex of the parabola is at (0, 1). The slope of the line through (0, 1) and another point $(h, h^2 + 1)$ on the parabola is $\frac{(h^2 + 1) - 1}{h - 0} = h$. Since $\lim_{h\to 0} h = 0$, the slope of the line tangent to the parabola at its vertex is 0.

[redacted]

**7.** $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = (1-1)^2 = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x + 2) = 1 + 2 = 3$

[redacted]

**9.** No, the two one-sided limits are different (see Exercise 7).

[redacted]

## **Section 3.1** Exercises

**1.** $\frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

$=\lim_{h\to 0}\frac{-h}{x^2+hx}\frac{1}{h}$

$=\lim_{h\to 0}\frac{-h}{x^2h+h^2x}=\lim_{h\to 0}-\frac{1}{x^2+hx}$

$=-\frac{1}{x^2}-\frac{1}{(2)^2}=-\frac{1}{4}$

[redacted]

**3.** $\frac{d}{dx}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{3-(x+h)^2-(3-x^2)}{h}$

$=\lim_{h\to 0}\frac{-2xh-h^2}{h}$

$=\lim_{h\to 0}-2x-h=-2x$

$-2(-1)=2$

[redacted]

**5.** $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$=\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}$

$=\lim_{x\to a}\frac{a-x}{ax}\left(\frac{1}{x-a}\right)$

$=\lim_{x\to a}-\frac{1}{ax}=-\frac{1}{a^2}$

$-\frac{1}{(2)^2}=-\frac{1}{4}$

[redacted]

**7.** $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$=\lim_{x\to a}\frac{\sqrt{x+1}-\sqrt{a+1}}{x-a}$

$=\lim_{x\to a}\frac{x-a}{x-a(\sqrt{x+1}+\sqrt{a+1})}$

$=\lim_{x\to a}\frac{1}{2\sqrt{a+1}}=\frac{1}{2\sqrt{3+1}}$

$=\frac{1}{4}$

[redacted]

**9.** $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{[3(x+h)-12]-(3x-12)}{h}$

$=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3$

[redacted]

**11.** Let $f(x)=x^2$

$\frac{d}{dx}(x^2)=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$

$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$

$=\lim_{h\to 0}(2x+h)=2x$

[redacted]

13. The graph of $y = f_1(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so its derivative is negative for $x < 0$ and positive for $x > 0$. (b)

[redacted]

15. The graph of $y = f_3(x)$ oscillates between increasing and decreasing, so its derivative oscillates between positive and negative. (d)

[redacted]

17. (a) The tangent line has slope 5 and passes through (2, 3).

$$y = 5(x-2)+3$$
$$y = 5x-7$$

(b) The normal line has slope $-\frac{1}{5}$ and passes through (2, 3).

$$y = -\frac{1}{5}(x-2)+3$$
$$y = -\frac{1}{5}x+\frac{17}{5}$$

[redacted]

19. Let $f(x) = x^3$.

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0}\frac{(1+h)^3-1^3}{h}$$
$$= \lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h}$$
$$= \lim_{h\to 0}(3+3h+h^2) = 3$$

(a) The tangent line has slope 3 and passes through (1, 1). Its equation is $y = 3(x-1)+1$, or $y = 3x-2$.

(b) The normal line has slope $-\frac{1}{3}$ and passes through (1, 1).

Its equation is $y = -\frac{1}{3}(x-1)+1$, or $y = -\frac{1}{3}x+\frac{4}{3}$.

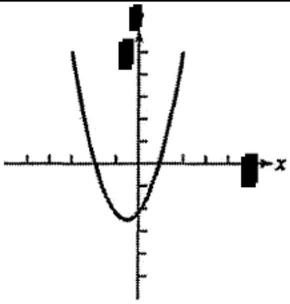
[redacted]

21. (a) The amount of daylight is increasing at the fastest rate when the slope of the graph is largest. This occurs about one-fourth of the way through the year, sometime around April 1. The rate at this time is approximately $\frac{4 \text{ hours}}{24 \text{ days}}$ or $\frac{1}{6}$ hour per day.

(b) Yes, the rate of change is zero when the tangent to the graph is horizontal. This occurs near the beginning of the year and halfway through the year, around January 1 and July 1.

(c) Positive: January 1 through July 1
Negative: July 1 through December 31

[redacted]




**23. (a)** Using Figure 3.10a, the number of rabbits is largest after 40 days and smallest from about 130 to 200 days. Using Figure 3.10b, the derivative is 0 at these times.

**(b)** Using Figure 3.10b, the derivative is largest after 20 days and smallest after about 63 days. Using Figure 3.10a, there were 1700 and about 1300 rabbits, respectively, at these times.

[redacted]

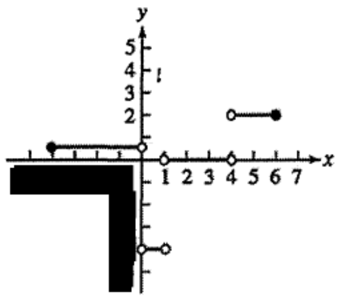
**25.** Each of the functions $y = \sin x,\ y = x,\ y = \sqrt{x}$ has the property that $y(0) = 0$ but the graph has nonzero slope (or undefined slope) at $x = 0$, so none of these functions can be its own derivative. The function $y = x^2$ is not its own derivative because $y(1) = 1$ but

$$y'(1) = \lim_{h\to 0}\frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2.$$

This leaves only $e^x$, which can plausibly be its own derivative because both the function value and the slope increase from very small positive values to very large values as we move from left to right along the graph. (iv)

[redacted]

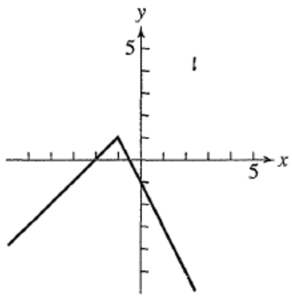
[redacted]



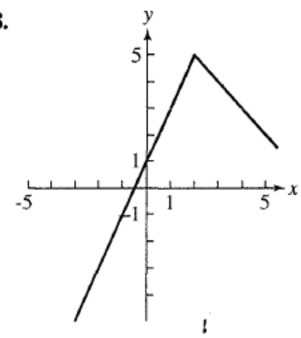

[redacted]

**27.** For $x > -1$, the graph of $y = f(x)$ must lie on a line of slope $-2$ that passes through $(0, -1)$: $y = -2x - 1$. Then $y(-1) = -2(-1) - 1 = 1$, so for $x < -1$, the graph of $y = f(x)$ must lie on a line of slope 1 that passes through $(-1, 1)$: $y = 1(x + 1) + 1$ or $y = x + 2$.

Thus $f(x) = \begin{cases} x+2, & x < -1 \\ -2x-1, & x \ge -1 \end{cases}$



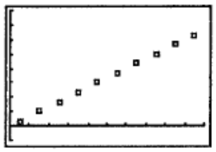

**28.**



**29.**

| Midpoint of Interval ($x$) | Slope $\left(\frac{\Delta y}{\Delta x}\right)$ |
|---|---|
| 0.5 | $\frac{3.3-0}{1-0}=3.3$ |
| 1.5 | $\frac{13.3-3.3}{2-1}=10.0$ |
| 2.5 | $\frac{29.9-13.3}{3-2}=16.6$ |
| 3.5 | $\frac{53.2-29.9}{4-3}=23.3$ |
| 4.5 | $\frac{83.2-53.2}{5-4}=30.0$ |
| 5.5 | $\frac{119.8-83.2}{6-5}=36.6$ |
| 6.5 | $\frac{163.0-119.8}{7-6}=43.2$ |
| 7.5 | $\frac{212.9-163.0}{8-7}=49.9$ |
| 8.5 | $\frac{269.5-212.9}{9-8}=56.6$ |
| 9.5 | $\frac{332.7-269.5}{10-9}=63.2$ |

A graph of the derivative data is shown.



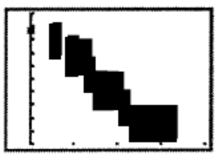
[0,10] by [−10,80]

**(a)** The derivative represents the speed of the skier.

**(b)** Since the distances are given in feet and the times are given in seconds, the units are feet per second.

**(c)** The graph appears to be approximately linear and passes through (0, 0) and (9.5, 63.2), so the slope is $\frac{63.2-0}{9.5-0}\approx 6.65$. The equation of the derivative is approximately $D = 6.65t$.

[redacted]



[redacted]

[redacted]

**31.** $f(x) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

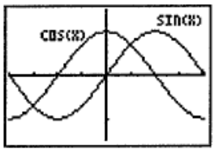
$= \lim_{h \to 0^+} \frac{3(1+h) - 2 - 2}{h}$

$= \lim_{h \to 0^+} \frac{3h - 1}{h} = -\infty$

Does not exist.

[redacted]

**33.**



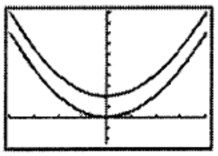

$[-\pi, \pi]$ by $[-1.5, 1.5]$

The cosine function could be the derivative of the sine function. The values of the cosine are positive where the sine is increasing, zero where the sine has horizontal tangents, and negative where sine is decreasing.

[redacted] $= \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$ [redacted]

**35.** Two parabolas are parallel if they have the same derivative at every value of $x$. This means that their tangent lines are parallel at each value of $x$.

Two such parabolas are given by $y = x^2$ and $y = x^2 + 4$. They are graphed below.



$[-4, 4]$ by $[-5, 20]$

The parabolas are "everywhere equidistant," as long as the distance between them is always measured along a vertical line.

[redacted]

**37.** False. Let $f(x) = \frac{|x|}{x}$. The left hand derivative at $x = 0$ is $-1$ and the right hand derivative at $x = 0$ is $1$. $f'(0)$ does not exist.

[redacted]

**39.** A. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{1 - 3(x+h)^2 - (1 - 3x^2)}{h}$

$= \lim_{h \to 0} \frac{-6xh - 3h^2}{h} = \lim_{h \to 0} 6x + h = -6x$

$-6(1) = -6.$

[redacted]

**41.** C. $\lim_{h \to 0^+} \frac{(2(0+h) - 1) - (2(0) - 1)}{h}$

$= \lim_{h \to 0^+} \frac{2h}{h} = 2$

[redacted]

[redacted]

**43.** The $y$-intercept of the derivative is $b - a$.

[redacted]

**45.** **(a)** $1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.992$

Alternate method: $\frac{{}_{365}P_3}{365^3} \approx 0.992$

**(b)** Using the answer to part (a), the probability is about $1 - 0.992 = 0.008$.

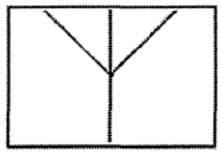
**(c)** Let $P$ represent the answer to part (b), $P \approx 0.008$. Then the probability that three people all have different birthdays is $1 - P$. Adding a fourth person, the probability that all have different birthdays is $(1-P)\left(\frac{362}{365}\right)$, so the probability of a shared birthday is $1-(1-P)\left(\frac{362}{365}\right) \approx 0.016$.

**(d)** No. Clearly February 29 is a much less likely birth date. Furthermore, census data do not support the assumption that the other 365 birth dates are equally likely. However, this simplifying assumption may still give us some insight into this problem even if the calculated probabilities aren't completely accurate.

## **Section 3.2** Differentiability (pp. 109–115)

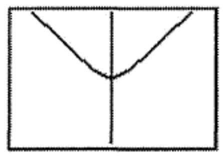
### Exploration 1 Zooming in to "See" Differentiability

**1.** Zooming in on the graph of $f$ at the point (0, 1) always produces a graph exactly like the one shown below, provided that a square window is used. The corner shows no sign of straightening out.



[−0.25, 0.25] by [0.836, 1.164]

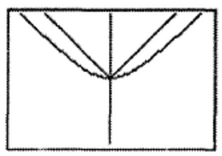
**2.** Zooming in on the graph of $g$ at the point (0, 1) begins to reveal a smooth turning point. This graph shows the result of three zooms, each by a factor of 4 horizontally and vertically, starting with the window. [−4, 4] by [−1.624, 3.624].



[−0.0625, 0.0625] by [0.959, 1.041]

**3.** On our grapher, the graph became horizontal after 8 zooms. Results can vary on different machines.

**4.** As we zoom in on the graphs of $f$ and $g$ together, the differentiable function gradually straightens out to resemble its tangent line, while the nondifferentiable function stubbornly retains its same shape.



[−0.03125, 0.03125] by [0.9795, 1.0205]

### Exploration 2 Looking at the Symmetric Difference Quotient Analytically

**1.** $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^2 - 10^2}{0.01} = 20.01$

$f'(10) = 2 \cdot 10 = 20$

The difference quotient is 0.01 away from $f'(10)$.

**2.** $\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^2 - (9.99)^2}{0.02} = 20$

The symmetric difference quotient exactly equals $f'(10)$.

**3.** $\frac{f(10+h)-f(10)}{h} = \frac{(10.01)^3 - 10^3}{0.01} = 300.3001$.

$f'(10) = 3 \cdot 10^2 = 300$

The difference quotient is 0.3001 away from $f'(10)$.

$\frac{f(10+h)-f(10-h)}{2h} = \frac{(10.01)^3 - (9.99)^3}{0.02} = 300.0001$. The symmetric difference quotient is 0.0001 away from $f'(10)$.

## Quick Review 3.2

1. Yes
2. No (The $f(h)$ term in the numerator is incorrect.)
3. Yes
4. Yes
5. No (The denominator for this expression should be $2h$).
6. All reals
7. $[0, \infty)$
8. $[3, \infty)$
9. The equation is equivalent to $y = 3.2x + (3.2\pi + 5)$, so the slope is 3.2.
10. $\dfrac{f(3+0.001)-f(3-0.001)}{0.002} = \dfrac{5(3+0.001)-5(3-0.001)}{0.002}$
$$= \frac{5(0.002)}{0.002} = 5$$

## Section 3.2 Exercises

1. Left-hand derivative:
$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{h^2-0}{h} = \lim_{h\to 0^-} h = 0$$
Right-hand derivative:
$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1$$
Since $0 \neq 1$, the function is not differentiable at the point $P$.

[redacted]

3. Left-hand derivative:
$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{\sqrt{1+h}-1}{h}$$
$$= \lim_{h\to 0^-}\frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0^-}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0^-}\frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$
Right-hand derivative:
$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{[2(1+h)-1]-1}{h}$$
$$= \lim_{h\to 0^+}\frac{2h}{h}$$
$$= \lim_{h\to 0^+} 2 = 2$$
Since $\frac{1}{2} \neq 2$, the function is not differentiable at the point $P$.

[redacted]

5. (a) All points in $[-3, 2]$
(b) None
(c) None

[redacted]

7. (a) All points in $[-3, 3]$ except $x = 0$
(b) None
(c) $x = 0$

[redacted]

9. (a) All points in $[-1, 2]$ except $x = 0$
(b) $x = 0$
(c) None

[redacted]

11. Since $\lim_{x\to 0} \tan^{-1} x = \tan^{-1} 0 = 0 \neq y(0)$, the problem is a discontinuity.

[redacted]

13. Note that $y = x + \sqrt{x^2} + 2 = x + |x| + 2$
$$= \begin{cases} 2, & x \le 0 \\ 2x+2, & x > 0. \end{cases}$$
$$\lim_{h\to 0^-}\frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-}\frac{2-2}{h} = \lim_{h\to 0^-} 0 = 0$$
$$\lim_{h\to 0^+}\frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+}\frac{(2h+2)-2}{h} = \lim_{h\to 0^+} 2 = 2$$
The problem is a corner.

[redacted]

**15.** Note that $y = 3x - 2|x| - 1 = \begin{cases} 5x-1, & x \le 0 \\ x-1, & x > 0 \end{cases}$

$$\lim_{h\to 0^-} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^-} \frac{(5h-1)-(-1)}{h} = \lim_{h\to 0^-} 5 = 5$$

$$\lim_{h\to 0^+} \frac{y(0+h)-y(0)}{h} = \lim_{h\to 0^+} \frac{(h-1)-(-1)}{h} = \lim_{h\to 0^+} 1 = 1$$

The problem is a corner.

[redacted]

**17.** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$

$= \frac{4(0.001)-(0.001)^2-(4(-0.001)-(-0.001)^2)}{0.002}$

$= 4$, yes it is differentiable.

[redacted]

**19.** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$

$= \frac{4(1.001)+(1.001)^2-4(0.999)-(0.99)^2}{0.002}$

$= 2$, yes it is differentiable.

[redacted]

**21.** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$

$= \frac{(-1.999)^3-4(-1.999)-((-2.001)^3-4(-2.001))}{0.002}$

$= 8.000001$, yes it is differentiable.

[redacted]

**23.** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$

$= \frac{(0.001)^{2/3}-(-0.001)^{2/3}}{0.002}$

$= 0$, no it is not differentiable.

[redacted]

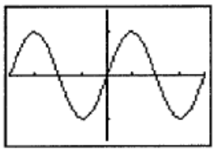
**25.** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$

$= \frac{(0.001)^{2/5}-(-0.001)^{2/5}}{0.002}$

$= 0$, no it is not differentiable.

[redacted]

**27.**

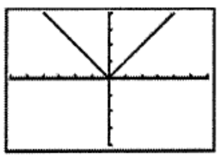


$[-2\pi, 2\pi]$ by $[-1.5, 1.5]$

$\frac{dy}{dx} = \sin x$

[redacted]

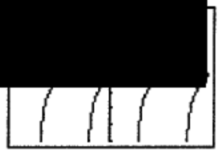
**29.**



$[-6, 6]$ by $[-4, 4]$

$\frac{dy}{dx} = \text{abs}(x)$ or $|x|$

[redacted]



[redacted]

**31.** Find the zeros of the denominator.

$x^2 - 4x - 5 = 0$
$(x+1)(x-5) = 0$
$x = -1$ or $x = 5$

The function is a rational function, so it is differentiabe for all $x$ in its domain: all reals except $x = -1, 5$.

[redacted]

**33.** Note that the sine function is odd, so

$$P(x) = \sin\left(|x|\right) - 1 = \begin{cases} -\sin x - 1, & x < 0 \\ \sin x - 1, & x \ge 0. \end{cases}$$

The graph of $P(x)$ has a corner at $x = 0$. The function is differentiable for all reals except $x = 0$.

[redacted]

**35.** The function is piecewise-defined in terms of polynomials, so it is differentiable everywhere except possibly at $x = 0$ and at $x = 3$. Check $x = 0$:

$$\lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)^2 - 1}{h} = \lim_{h\to 0^-} \frac{h^2 + 2h}{h}$$
$$= \lim_{h\to 0^-} (h+2) = 2$$
$$\lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^+} \frac{(2h+1) - 1}{h} = \lim_{h\to 0^+} 2 = 2$$

The function is differentiable at $x = 0$.

Check $x = 3$:

Since $g(3) = (4-3)^2 = 1$ and $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (2x+1) = 2(3) + 1 = 7$, the function is not continuous (and hence not differentiable) at $x = 3$. The function is differentiable for all reals except $x = 3$.

[redacted]

**37.** The function $f(x)$ does not have the intermediate value property. Choose some $a$ in $(-1, 0)$ and $b$ in $(0, 1)$. Then $f(a) = 0$ and $f(b) = 1$, but $f$ does not take on any value between 0 and 1. Therefore, by the Intermediate Value Theorem for Derivatives, $f$ cannot be the derivative of any function on $[-1, 1]$.

[redacted]

**39. (a)** $\lim_{x\to 1^-} f(x) = f(1)$

$\lim_{x\to 1^-} (3 - x) = a(1)^2 + b(1)$

$2 = a + b$

The relationship is $a + b = 2$.

**39. Continued**

**(b)** Since the function needs to be continuous, we may assume that $a + b = 2$ and $f(1) = 2$.

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{3-(1+h)-2}{h}$$
$$= \lim_{h\to 0^-} (-1) = -1$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{a(1+h)^2+b(1+h)-2}{h}$$
$$= \lim_{h\to 0^-} \frac{a+2ah+ah^2+b+bh-2}{h}$$
$$= \lim_{h\to 0^-} \frac{2ah+ah^2+bh+(a+b-2)}{h}$$
$$= \lim_{h\to 0^-} (2a+ah+b)$$
$$= 2a+b$$

Therefore, $2a + b = -1$. Substituting $2 - a$ for $b$ gives $2a + (2 - a) = -1$, so $a = -3$.
Then $b = 2 - a = 2 - (-3) = 5$. The values are $a = -3$ and $b = 5$.

[redacted]

**41.** False. The function $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

[redacted]

**43.** A. $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a-h)}{2h}$

$$= \frac{\sqrt[3]{1.001-1} - \sqrt[3]{0.999-1}}{0.002}$$
$$= \infty$$
$$f(1) = \sqrt[3]{1-1} = 0$$

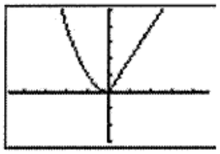
[redacted]

**45.** C. $\lim\limits_{h\to 0} \dfrac{(0+h)^2+1-(0^2+1)}{h}$

$$= \lim_{h\to 0} \frac{h^2}{h} = 0$$

[redacted]

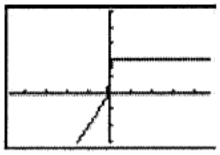
**47. (a)**



[−4.7, 4.7] by [−3, 5]

**(b)** See exersize 46.

**(c)**



[−4.7, 4.7] by [−3, 5]

**(d)** NDER (Y1, $x$, –0.1) = –0.1, NDER(Y1, $x$, 0) = 0.9995, NDER (Y1, $x$, 0.1) = 2.

[redacted]

## **Section 3.3** Rules for Differentiation (pp. 116–126)

### Quick Review 3.3

**1.** $(x^2-2)(x^{-1}+1) = x^2x^{-1} + x^2 \cdot 1 - 2x^{-1} - 2 \cdot 1$
$= x + x^2 - 2x^{-1} - 2$

**2.** $\left(\dfrac{x}{x^2+1}\right)^{-1} = \dfrac{x^2+1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = x + x^{-1}$

**3.** $3x^2 - \dfrac{2}{x} + \dfrac{5}{x^2} = 3x^2 - 2x^{-1} + 5x^{-2}$

**4.** $\dfrac{3x^4 - 2x^3 + 4}{2x^2} = \dfrac{3x^4}{2x^2} - \dfrac{2x^3}{2x^2} + \dfrac{4}{2x^2}$

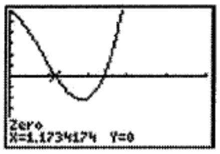
$= \dfrac{3}{2}x^2 - x + 2x^{-2}$

**5.** $(x^{-1}+2)(x^{-2}+1) = x^{-1}x^{-2} + x^{-1} \cdot 1 + 2x^{-2} + 2 \cdot 1$

$= x^{-3} + x^{-1} + 2x^{-2} + 2$

**6.** $\dfrac{x^{-1} + x^{-2}}{x^{-3}} = x^3(x^{-1} + x^{-2}) = x^2 + x$

**7.**




[0, 5] by [−6, 6]

At $x \approx 1.173$, $500x^6 \approx 1305$.
At $x \approx 2.394$, $500x^6 \approx 94{,}212$.
After rounding, we have:
At $x \approx 1$, $500x^6 \approx 1305$.
At $x \approx 2$, $500x^6 \approx 94{,}212$.

**8. (a)** $f(10) = 7$

**(b)** $f(0) = 7$

**(c)** $f(x+h) = 7$

**(d)** $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x\to a} \dfrac{7-7}{x-a} = \lim\limits_{x\to a} 0 = 0$

**9.** These are all constant functions, so the graph of each function is a horizontal line and the derivative of each function is 0.

**10. (a)** $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{x+h}{\pi} - \frac{x}{\pi}}{h}$

$= \lim\limits_{h\to 0} \dfrac{x+h-x}{\pi h} = \lim\limits_{h\to 0} \dfrac{1}{\pi} = \dfrac{1}{\pi}$

**(b)** $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{\pi}{x+h} - \frac{\pi}{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\pi x - \pi(x+h)}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-\pi h}{hx(x+h)}$

$= \lim\limits_{h\to 0} -\dfrac{\pi}{x(x+h)} = -\dfrac{\pi}{x^2} = -\pi x^{-2}$

## Section 3.3 Exercises

**1.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(-x^2) + \dfrac{d}{dx}(3) = -2x + 0 = -2x$

[redacted]

**3.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x) + \dfrac{d}{dx}(1) = 2 + 0 = 2$

[redacted]

**5.** $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) + \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right) + \dfrac{d}{dx}(x)$

$= x^2 + x + 1$

[redacted]

**7.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3 - 2x^2 + x + 1)$

$= 3x^2 - 4x + 1 = 0$

$x = \dfrac{1}{3}, 1$

[redacted]

**9.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 4x^2 + 1)$

$= 4x^3 - 8x = 0$

$x = 0, \pm\sqrt{2}$

[redacted]

**11.** $\dfrac{dy}{dx} = \dfrac{d}{dx}(5x^3 - 3x^5)$

$= 15x^2 - 15x^4 = 0$

$x = -1, 0, 1$

[redacted]

**13. (a)** $\dfrac{dy}{dx} = \dfrac{d}{dx}[(x+1)(x^2+1)]$

$= (x+1)\dfrac{d}{dx}(x^2+1) + (x^2+1)\dfrac{d}{dx}(x+1)$

$= (x+1)(2x) + (x^2+1)(1)$

$= 2x^2 + 2x + x^2 + 1$

$= 3x^2 + 2x + 1$

**13. Continued**

**(b)** $\frac{dy}{dx} = \frac{d}{dx}[(x+1)(x^2+1)]$

$= \frac{d}{dx}(x^3 + x^2 + x + 1)$

$= 3x^2 + 2x + 1$

[redacted]

**15.** $(x^3 + x + 1)(x^4 + x^2 + 1)$

$\frac{d}{dx}(x^7 + 2x^5 + x^4 + 2x^3 + x^2 + x + 1)$

$= 7x^6 + 10x^4 + 4x^3 + 6x^2 + 2x + 1$

[redacted]

**17.** $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+5}{3x-2} = \frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = -\frac{19}{(3x-2)^2}$

[redacted]

**19.** $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{(x-1)(x^2+x+1)}{x^3}\right) = \frac{d}{dx}\left(\frac{x^3-1}{x^3}\right)$

$= \frac{d}{dx}(1 - x^{-3}) = 0 + 3x^{-4} = \frac{3}{x^4}$

[redacted]

**21.** $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{1-x^3}\right) = \frac{(1-x^3)(2x) - x^2(-3x^2)}{(1-x^3)^2} = \frac{x^4 + 2x}{(1-x^3)^2}$

[redacted]

**23. (a)** At $x = 0$, $\frac{d}{dx}(uv) = u(0)v'(0) + v(0)u'(0)$

$= (5)(2) + (-1)(-3) = 13$

**(b)** At $x = 0$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(0)u'(0) - u(0)v'(0)}{[v(0)]^2}$

$= \frac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$

**(c)** At $x = 0$, $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u(0)v'(0) - v(0)u'(0)}{[u(0)]^2}$

$= \frac{(5)(2)-(-1)(-3)}{(5)^2} = \frac{7}{25}$

**(d)** At $x = 0$, $\frac{d}{dx}(7v - 2u) = 7v'(0) - 2u'(0)$

$= 7(2) - 2(-3) = 20$

[redacted]

**25.** $y'(x) = 2x + 5$

$y'(3) = 2(3) + 5 = 11$

The slope is 11. (iii)

[redacted]

**27.** $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3+1}{2x}\right)$

$= \frac{(3x^2)2x - 2(x^3+1)}{4x^2}$

$= \frac{4x^3-2}{4x^2}$

$y'(1) = \frac{4(1)^3-2}{4(1)^2} = \frac{1}{2}$

$y(1) = \frac{(1)^3+1}{2(1)} = 1$

$y = \frac{1}{2}(x-1)+1 = \frac{1}{2}x + \frac{1}{2}$

[redacted]

**29.** $\frac{dy}{dx} = \frac{d}{dx}(4x^{-2} - 8x + 1)$

$= -8x^{-3} - 8$

[redacted]

**31.** $\frac{dy}{dx} = \frac{d}{dx}\frac{\sqrt{x}-1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$

[redacted]

**33.** $y = x^4 + x^3 - 2x^2 + x - 5$

$y^{\mathrm{I}} = 4x^3 + 3x^2 - 2x + 1$

$y^{\mathrm{II}} = 12x^2 + 6x - 2$

$y^{\mathrm{III}} = 24x + 6$

$y^{\mathrm{IV}} = 24$

[redacted]

**35.** $y = x^{-1} + x^2$

$y^{\mathrm{I}} = -x^{-2} + 2x$

$y^{\mathrm{II}} = 2x^{-3} + 2$

$y^{\mathrm{III}} = -6x^{-4}$

$y^{\mathrm{IV}} = -24x^{-5}$

[redacted]

**37.** $y'(x) = 3x^2 - 3$

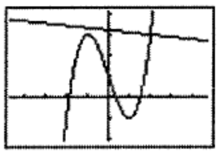
$y'(2) = 3(2)^2 - 3 = 9$

The tangent line has slope 9, so the perpendicular line has slope $-\frac{1}{9}$ and passes through (2, 3).

$y = -\frac{1}{9}(x-2) + 3$

$y = -\frac{1}{9}x + \frac{29}{9}$

Graphical support:



[−4.7, 4.7] by [−2.1, 4.1]

[redacted]

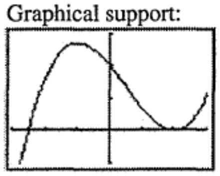
**39.** $y'(x) = 6x^2 - 6x - 12$

$$= 6(x^2 - x - 2)$$

$$= 6(x+1)(x-2)$$

The tangent is parallel to the $x$-axis when $y' = 0$, at $x = -1$ and at $x = 2$. Since $y(-1) = 27$ and $y(2) = 0$, the two points where this occurs are $(-1, 27)$ and $(2, 0)$.

Graphical support:



$[-3, 3]$ by $[-10, 30]$

[redacted]

**41.** $y'(x) = \dfrac{(x^2+1)(4) - 4x(2x)}{(x^2+1)^2} = \dfrac{-4x^2+4}{(x^2+1)^2}$

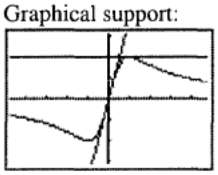
At the origin: $y'(0) = 4$

The tangent is $y = 4x$.

At $(1, 2)$: $y'(1) = 0$

The tangent is $y = 2$.

Graphical support:



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

[redacted]

**43. (a)** Let $f(x) = x$.

$$\frac{d}{dx}(x) = f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h) - x}{h}$$

$$= \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0}(1) = 1$$

**(b)** Note that $u = u(x)$ is a function of $x$.

$$\frac{d}{dx}(-u) = \lim_{h\to 0}\frac{-u(x+h) - [-u(x)]}{h}$$

$$= \lim_{h\to 0}\left(-\frac{u(x+h) - u(x)}{h}\right)$$

$$= -\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} = -\frac{du}{dx}$$

[redacted]

**45.** $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{f(x)\cdot 0 - 1\cdot \dfrac{d}{dx}f(x)}{[f(x)]^2} = -\dfrac{f'(x)}{[f(x)]^2}$

[redacted]

**47.** $\dfrac{ds}{dt} = \dfrac{d}{dt}(4.9t^2) = 9.8t$

$$\frac{d^2s}{dt^2} = \frac{d}{dt}(9.8t) = 9.8$$

[redacted]

**49.** If the radius of a circle is changed by a very small amount $\Delta r$, the change in the area can be thought of as a very thin strip with length given by the circumference, $2\pi r$, and width $\Delta r$. Therefore, the change in the area can be thought of as $(2\pi r)(\Delta r)$, which means that the change in the area divided by the change in the radius is just $2\pi r$.

[redacted]

**51.** Let $t(x)$ be the number of trees and $y(x)$ be the yield per tree $x$ years from now. Then $t(0)=156$, $y(0)=12$, $t'(0)=13$, and $y'(0)=1.5$. The rate of increase of production is

$$\frac{d}{dx}(ty)=t(0)y'(0)+y(0)t'(0)=(156)(1.5)+(12)(13)=390$$

bushels of annual production per year.

[redacted]

**53.** False. $\pi$ is a constant so $\frac{d}{dx}(\pi)=0$.

[redacted]

**55.** B. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

$$=(2)(1)+(-1)(3)$$
$$=-1$$

[redacted]

**57.** E. $\frac{d}{dx}\left(\frac{x+1}{x^2-1}\right)$

$$=\frac{(x-1)-(x+1)}{(x-1)^2}$$
$$=\frac{-2}{(x-1)^2}$$

[redacted]

**59. (a)** It is insignificant in the limiting case and can be treated as zero (and removed from the expression).

**(b)** It was "rejected" because it is incomparably smaller than the other terms: $v\,du$ and $u\,dv$.

**(c)** $\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$. This is equivalent to the product rule given in the text.

**(d)** Because $dx$ is "infinitely small," and this could be thought of as dividing by zero.

**(e)** $d\left(\frac{u}{v}\right)=\frac{u+du}{v+dv}-\frac{u}{v}$

$$=\frac{(u+du)(v)-(u)(v+dv)}{(v+dv)(v)}$$
$$=\frac{uv+vdu-uv-udv}{v^2+vdv}$$
$$=\frac{vdu-udv}{v^2}$$

## Quick Quiz Sections 3.1–3.3

**1.** D.

**2.** A. Since $\frac{dy}{dx}$ gives the slope:

$$m_1=\frac{1-2}{-1-1}=\frac{1}{2}$$
$$m_2=-\frac{1}{m_1}=-2$$

**3.** C. $\frac{dy}{dx}=\frac{d}{dx}\frac{4x-3}{2x+1}$

$$=\frac{4(2x+1)-2(4x-3)}{(2x-1)^2}$$
$$=\frac{10}{(2x-1)^2}$$

**4. (a)** $\frac{dy}{dx}=\frac{d}{dx}(x^4-4x^2)$

$$=4x^2-8x=0$$
$$x=0,\ \pm\sqrt{2}$$

**(b)** $x=1 \qquad y=(1)^4-4(1)^2$

$$y=-3$$
$$y=m(x-x_1)+y_1$$
$$y=-4(x-1)-3$$
$$y=-4x+1$$

**4. Continued**

(c) $m_2 = -\frac{1}{m_1} = \frac{1}{4}$

$$y = \frac{1}{4}(x-1) - 3$$
$$= \frac{1}{4}x - \frac{1}{4} - 3 = \frac{1}{4}x - \frac{13}{4}$$

## Section 3.4 Velocity and Other Rates of Change (pp. 127–140)

### Exploration 1 Growth Rings on a Tree

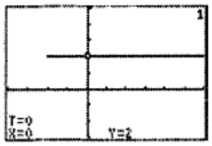
1. Figure 3.22 is a better model, as it shows rings of equal *area* as opposed to rings of equal *width*. It is not likely that a tree could sustain increased growth year after year, although climate conditions do produce some years of greater growth than others.

2. Rings of equal area suggest that the tree adds approximately the same amount of wood to its girth each year. With access to approximately the same raw materials from which to make the wood each year, this is how most trees actually grow.

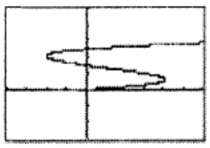
3. Since change in area is constant, so also is $\frac{\text{change in area}}{2\pi}$.
If we denote this latter constant by $k$, we have
$\frac{k}{\text{change in radius}} = r$, which means that $r$ varies inversely as the change in the radius. In other words, the change in radius must get smaller when $r$ gets bigger, and vice-versa.

### Exploration 2 Modeling Horizontal Motion

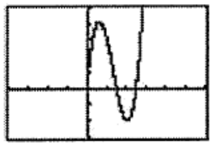
1. The particle reverses direction at about $t = 0.61$ and $t = 2.06$.



2. When the trace cursor is moving to the right the particle is moving to the right, and when the cursor is moving to the left the particle is moving to the left. Again we find the particle reverses direction at about $t = 0.61$ and $t = 2.06$.



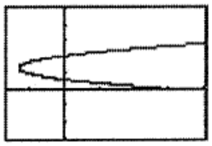
3. When the trace cursor is moving upward the particle is moving to the right, and when the cursor is moving downward the particle is moving to the left. Again we find the same values of $t$ for when the particle reverses direction.



4. We can represent the velocity by graphing the parametric equations
$x_4(t) = x_1'(t) = 12t^2 - 32t + 15,\ y_4(t) = 2$ (part 1),
$x_5(t) = x_1'(t) = 12t^2 - 32t + 15,\ y_5(t) = t$ (part 2),
$x_6(t) = t,\ y_6(t) = x_1'(t) = 12t^2 - 32t + 15$ (part 3)

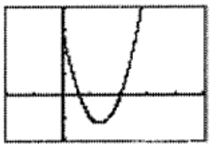
[−8, 20] by [−3, 5]

$(x_4, y_4)$



[−8, 20] by [−3, 5]

$(x_5, y_5)$



[−2, 5] by [−10, 20]

$(x_6, y_6)$

For $(x_4, y_4)$ and $(x_5, y_5)$, the particle is moving to the right when the $x$-coordinate of the graph (velocity) is positive, moving to the left when the $x$-coordinate of the graph (velocity) is negative, and is stopped when the $x$-coordinate of the graph (velocity) is 0. For $(x_6, y_6)$,the particle is moving to the right when the $y$-coordinate of the graph (velocity) is positive, moving to the left when the $y$-coordinate of the graph (velocity) is negative, and is stopped when the $y$-coordinate of the graph (velocity) is 0.

### Exploration 3 Seeing Motion on a Graphing Calculator

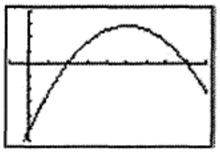
1. Let $t$Min = 0 and $t$Max = 10.

2. Since the rock achieves a maximum height of 400 feet, set $y$Max to be slightly greater than 400, for example $y$Max = 420.

**4.** The grapher proceeds with constant increments of $t$ (time), so pixels appear on the screen at regular time intervals. When the rock is moving more slowly, the pixels appear closer together. When the rock is moving faster, the pixels appear farther apart. We observe faster motion when the pixels are farther apart.

## Quick Review 3.4

**1.** The coefficient of $x^2$ is negative, so the parabola opens downward.
Graphical support:



[−1, 9] by [−300, 200]

**2.** The $y$-intercept is $f(0) = -256$.
See the solution to Exercise 1 for graphical support.

**3.** The $x$-intercepts occur when $f(x) = 0$.

$$-16x^2 + 160x - 256 = 0$$
$$-16(x^2 - 10x + 16) = 0$$
$$-16(x-2)(x-8) = 0$$
$$x = 2 \text{ or } x = 8$$

The $x$-intercepts are 2 and 8. See the solution to Exercise 1 for graphical support.

**4.** Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x-5)^2 + 144$, the range is $(-\infty, 144]$.
See the solution to Exercise 1 for graphical support.

**5.** Since $f(x) = -16(x^2 - 10x + 16)$
$= -16(x^2 - 10x + 25 - 9) = -16(x-5)^2 + 144$, the vertex is at $(5, 144)$. See the solution to Exercise 1 for graphical support.

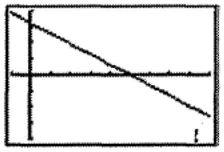
**6.**
$$f(x) = 80$$
$$-16x^2 + 160x - 256 = 80$$
$$-16x^2 + 160x - 336 = 0$$
$$-16(x^2 - 10x + 21) = 0$$
$$-16(x-3)(x-7) = 0$$
$$x = 3 \text{ or } x = 7$$

$f(x) = 80$ at $x = 3$ and at $x = 7$.
See the solution to Exercise 1 for graphical support.

**7.**
$$\frac{dy}{dx} = 100$$
$$-32x + 160 = 100$$
$$60 = 32x$$
$$x = \frac{15}{8}$$

$\frac{dy}{dx} = 100$ at $x = \frac{15}{8}$

Graphical support: the graph of NDER $f(x)$ is shown.



[−1, 9] by [−200, 200]

**8.**
$$\frac{dy}{dx} > 0$$
$$-32x + 160 > 0$$
$$-32x > -160$$
$$x < 5$$

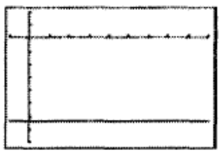
$\frac{dy}{dx} > 0$ when $x < 5$.
See the solution to Exercise 7 for graphical support.

**9.** Note that $f'(x) = -32x + 160$.

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = f'(3) = -32(3) + 160 = 64$$

For graphical support, use the graph shown in the solution to Exercise 7 and observe that NDER $(f(x), 3) \approx 64$.

**10.** $f'(x) = -32x + 160$
$f''(x) = -32$
At $x = 7$ (and, in fact, at any otherof $x$),
$\frac{d^2y}{dx^2} = -32.$
Graphical support: the graph of NDER(NDER $f(x)$) is shown.



[−1, 9] by [−40, 10]

## Section 3.4 Exercises

**1. (a)** $V(s) = s^3$

**(b)** $\frac{dv}{ds} = 3s^2$

**(c)** $V'(1) = 3(1)^2 = 3$
$V'(5) = 3(5)^2 = 75$

**(d)** $\frac{\text{in}^3}{\text{in}}$.

[redacted]

[redacted]

**3. (a)** $A(s) = \frac{\sqrt{3}}{4}s^2$

**(b)** $\frac{dA}{ds} = \frac{\sqrt{3}}{2}s$

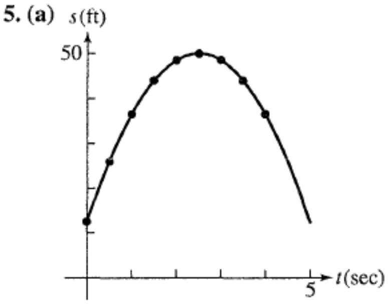
**(c)** $A'(2) = \frac{\sqrt{3}}{2}(2) = \sqrt{3}$

$A'(10) = \frac{\sqrt{3}}{2}(10) = 5\sqrt{3}$

**(d)** $\frac{\text{in}^2}{\text{in}}$.

[redacted]

**5. (a)**

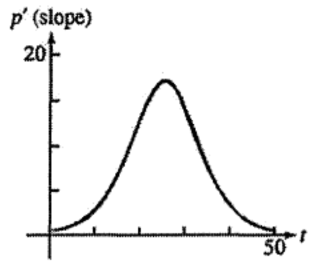



**(b)** $s'(1) = 18$, $s'(2.5) = 0$, $s'(3.5) = -12$

[redacted]

**7. (a)** We estimate the slopes at several points as follows, then connect the points to create a smooth curve.

| $t$ (days) | 0 | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| Slope (flies/ day) | 0.5 | 3.0 | 13.0 | 14.0 | 3.5 | 0.5 |




Horizontal axis: Days
Vertical axis: Flies per day

**(b)** Fastest: Around the 25th day
Slowest: Day 50 or day 0

[redacted]

**9. (a)** The particle moves forward when $v > 0$, for $0 \le t < 1$ and for $5 < t < 7$.
The particle moves backward when $v < 0$, for $1 < t < 5$.
The particle speeds up when $v$ is negative and decreasing, for $1 < t < 2$, and when $v$ is positive and increasing, for $5 < t < 6$.
The particle slows down when $v$ is positive and decreasing, for $0 \le t < 1$ and for $6 < t < 7$, and when $v$ is negative and increasing, for $3 < t < 5$.

**(b)** Note that the acceleration $a = \frac{dv}{dt}$ is undefined at $t = 2$, $t = 3$, and $t = 6$.
The acceleration is positive when $v$ is increasing, for $3 < t < 6$.
The acceleration is negative when $v$ is decreasing, for $0 \le t < 2$ and for $6 < t < 7$.
The acceleration is zero when $v$ is constant, for $2 < t < 3$ and for $7 < t \le 9$.

**(c)** The particle moves at its greatest speed when $|v|$ is maximized, at $t = 0$ and for $2 < t < 3$.

**9. Continued**

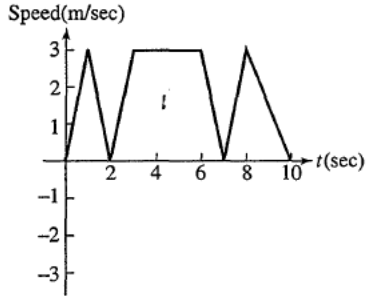
**(d)** The particle stands still for more than an instant when $v$ stays at zero, for $7 < t \le 9$.

[redacted]

**11. (a)** The body reverses direction when $v$ changes sign, at $t = 2$ and at $t = 7$.

**(b)** The body is moving at a constant speed, $|v| = 3$ m/sec, between $t = 3$ and $t = 6$.

**(c)** The speed graph is obtained by reflecting the negative portion of the velocity graph, $2 < t < 7$, about the $x$-axis.



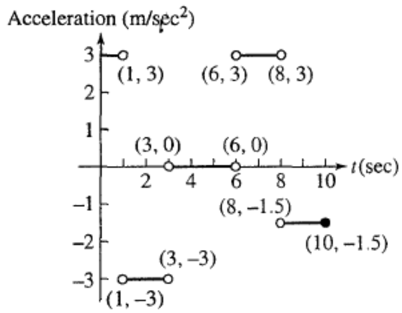

**(d)** For $0 \le t < 1$: $a = \dfrac{3-0}{1-0} = 3$ m/sec$^2$

For $1 < t < 3$: $a = \dfrac{-3-3}{3-1} = -3$ m/sec$^2$

For $3 < t < 6$: $a = \dfrac{-3-(-3)}{6-3} = 0$ m/sec$^2$

For $6 < t < 8$: $a = \dfrac{3-(-3)}{8-6} = 3$ m/sec$^2$

For $8 < t \le 10$: $a = \dfrac{0-3}{10-8} = -1.5$ m/sec$^2$




[redacted]

**13. (a)** Velocity: $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(24t - 0.8t^2) = 24 - 1.6t$ m/sec

Acceleration: $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(24 - 1.6t) = -1.6$ m/sec$^2$

**13. Continued**

**(b)** The rock reaches its highest point when $v(t) = 24 - 1.6t = 0$, at $t = 15$. It took 15 seconds.

**(c)** The maximum height was $s(15) = 180$ meters.

**(d)**
$$s(t) = \frac{1}{2}(180)$$
$$24t - 0.8t^2 = 90$$
$$0 = 0.8t^2 - 24t + 90$$
$$t = \frac{24 \pm \sqrt{(-24)^2 - 4(0.8)(90)}}{2(0.8)}$$
$$\approx 4.393,\ 25.607$$

It took about 4.393 seconds to reach half its maximum height.

**(e)**
$$s(t) = 0$$
$$24t - 0.8t^2 = 0$$
$$0.8t(30 - t) = 0$$
$$t = 0 \text{ or } t = 30$$

The rock was aloft from $t = 0$ to $t = 30$, so it was aloft for 30 seconds.

[redacted]

**15.** The rock reaches its maximum height when the velocity $s'(t) = 24 - 9.8t = 0$, at $t \approx 2.449$. Its maximum height is about $s(2.449) \approx 29.388$ meters.

[redacted]

**17.** The following is one way to simulate the problem situation.

For the moon:

$x_1(t) = 3(t < 160) + 3.1(t \geq 160)$

$y_1(t) = 832t - 2.6t^2$

$t$-values: 0 to 320

window: [0, 6] by [−10,000, 70,000]

For the earth:

$x_1(t) = 3(t < 26) + 3.1(t \geq 26)$

$y_1(t) = 832t - 16t^2$

$t$-values: 0 to 52

window: [0, 6] by [−1000, 11,000]

[redacted]

**19. (a)** Displacement: $= s(5) - s(0) = 12 - 2 = 10$ m

**(b)** Average velocity $= \dfrac{10 \text{ m}}{5 \text{ sec}} = 2$ m/sec

**(c)** Velocity $= s'(t) = 2t - 3$

At $t = 4$, velocity $= s'(4) = 2(4) - 3 = 5$ m/sec

**(d)** Acceleration $= s''(t) = 2$ m/sec$^2$

**(e)** The particle changes direction when $s'(t) = 2t - 3 = 0$, so $t = \dfrac{3}{2}$ sec.

**(f)** Since the acceleration is always positive, the position $s$ is at a minimum when the particle changes direction, at $t = \dfrac{3}{2}$ sec. Its position at this time is $s\left(\dfrac{3}{2}\right) = -\dfrac{1}{4}$ m.

[redacted]

**21. (a)** $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(t-2)^2(t-4)$
$$= (t-2)(3t-10)$$

**(b)** $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(t-2)(3t-10)$

$a(t) = 6t - 16$

**21. Continued**

**(c)** $v(t) = (t-2)(3t-10) = 0$

$t = 2, \frac{10}{3}$

**(d)** The particle starts at the point $s = -16$ when $t = 0$ and move right until it stops at $s = 0$ when $t = 2$, then it moves left to the point $s = -1.185$ when $t = \frac{10}{3}$ where it stops again, and finally continues right from there on.

[redacted]

**23.** $v(t) = s'(t) = 3t^2 - 12t + 9$

$a(t) = v'(t) = 6t - 12$

Find when velocity is zero.

$3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-1)(t-3) = 0$

$t = 1$ or $t = 3$

At $t = 1$, the acceleration is $a(1) = -6$ m/sec$^2$

At $t = 3$, the acceleration is $a(3) = 6$ m/sec$^2$

[redacted]

**25. (a)** $\frac{dy}{dt} = \frac{d}{dt}\left[6\left(1 - \frac{t}{12}\right)^2\right]$

$= \frac{d}{dt}\left[6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right)\right]$

$= \frac{d}{dt}\left(6 - t + \frac{1}{24}t^2\right)$

$= 0 - 1 + \frac{t}{12} = \frac{t}{12} - 1$

**(b)** The fluid level is falling fastest when $\frac{dy}{dt}$ is the most negative, at $t = 0$, when $\frac{dy}{dt} = -1$. The fluid level is falling slowest at $t = 12$, when $\frac{dy}{dt} = 0$.

**(c)**

$y$ is decreasing and $\frac{dy}{dt}$ is negative over the entire interval $y$ decreases more rapidly early in the interval, and the magnitude of $\frac{dy}{dt}$ is larger then. $\frac{dy}{dt}$ is 0 at $t = 12$, where the graph of $y$ seems to have a horizontal tangent.

[redacted]

[redacted]

27. (a) Average cost $= \frac{c(100)}{100} = \frac{11,000}{100} = \$110$ per machine

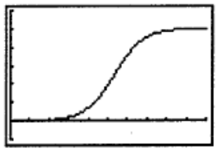
(b) $c'(x) = 100 - 0.2x$

Marginal cost $= c'(100) = \$80$ per machine

(c) Actual cost of 101st machine is $c(101) - c(100) = \$79.90$, which is very close to the marginal cost calculated in part (b).

[redacted]

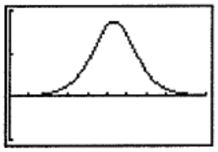
29. (a)



[0, 200] by [–2, 12]

(b) The values of $x$ which make sense are the whole numbers, $x \geq 0$.

(c)



[0, 200] by [–0.1, 0.2]

$P$ is most sensitive to changes in $x$ when $|P'(x)|$ is largest. It is relatively sensitive to changes in $x$ between approximately $x = 60$ and $x = 160$.

(d) The marginal profit, $P'(x)$, is greatest at $x = 106.44$. Since $x$ must be an integer, $P(106) \approx 4.924$ thousand dollars or \$4924.

(e) $P'(50) \approx 0.013$, or \$13 per package sold

$P'(100) \approx 0.165$, or \$165 per package sold

$P'(125) \approx 0.118$, or \$118 per package sold

$P'(150) \approx 0.031$, or \$31 per package sold

$P'(175) \approx 0.006$, or \$6 per package sold

$P'(300) \approx 10^{-6}$, or \$0.001 per package sold

(f) The limit is 10. The maximum possible profit is \$10,000 monthly.

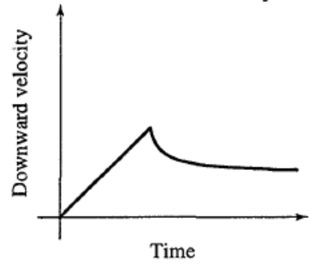
(g) Yes. In order to sell more and more packages, the company might need to lower the price to a point where they won't make any additional profit.

[redacted]

31. Graph C is position, graph A is velocity, and graph B is acceleration.
A is the derivative of C because it is positive, negative, and zero where C is increasing, decreasing, and has horizontal tangents, respectively. The relationship between B and A is similar.

32. Graph C is position, graph B is velocity, and graph A is acceleration.
B is the derivative of C because it is negative and zero where C is decreasing and has horizontal tangents, respectively.
A is the derivative of B because it is positive, negative, and zero where B is increasing, decreasing, and has horizontal tangents, respectively.

**33.** Note that "downward velocity" is positive when McCarthy is falling downward. His downward velocity increases steadily until the parachute opens, and then decreases to a constant downward velocity. One possible sketch:




[redacted]

**35.** Let $v_0$ be the exit velocity of a particle of lava. Then $s(t) = v_0 t - 16t^2$ feet, so the velocity is $\frac{ds}{dt} = v_0 - 32t$. Solving $\frac{ds}{dt} = 0$ gives $t = \frac{v_0}{32}$. Then the maximum height, in feet, is $s\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^{\,2}}{64}$. Solving $\frac{v_0^{\,2}}{64} = 1900$ gives $v_0 \approx \pm 348.712$. The exit velocity was about 348.712 ft/sec. Multiplying by $\frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}}$, we find that this is equivalent to about 237.758 mi/h.

[redacted]

**37.** The motion can be simulated in parametric mode using $x_1(t) = 2t^3 - 13t^2 + 22t - 5$ and $y_1(t) = 2$ in $[-6, 8]$ by $[-3, 5]$.

**(a)** It begins at the point $(-5, 2)$ moving in the positive direction. After a little more than one second, it has moved a bit past $(6, 2)$ and it turns back in the negative direction for approximately 2 seconds. At the end of that time, it is near $(-2, 2)$ and it turns back again in the positive direction. After that, it continues moving in the positive direction indefinitely, speeding up as it goes.

**(b)** The particle speeds up when its *speed* is increasing, which occurs during the approximate intervals $1.153 \le t \le 2.167$ and $t \ge 3.180$. It slows down during the approximate intervals $0 \le t \le 1.153$ and $2.167 \le t \le 3.180$. One way to determine the endpoints of these intervals is to use a grapher to find the minimums and maximums for the speed, $|\text{NDER } x(t)|$, using function mode in the window $[0, 5]$ by $[0, 10]$.

**(c)** The particle changes direction at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

**(d)** The particle is at rest "instantaneously" at $t \approx 1.153$ sec and at $t \approx 3.180$ sec.

**(e)** The velocity starts out positive but decreasing, it becomes negative, then starts to increase, and becomes positive again and continues to increase.
The speed is decreasing, reaches 0 at $t \approx 1.15$ sec, then increases until $t \approx 2.17$ sec, decreases until $t \approx 3.18$ sec when it is 0 again, and then increases after that.

**(f)** The particle is at (5, 2) when $2t^3 - 13t^2 + 22t - 5 = 5$ at $t \approx 0.745$ sec, $t \approx 1.626$ sec, and at $t \approx 4.129$ sec.

[redacted]

**39.** Since profit = revenue – cost, the Sum and Difference Rule gives $\frac{d}{dx}(\text{profit}) = \frac{d}{dx}(\text{revenue}) - \frac{d}{dx}(\text{cost})$, where $x$ is the number of units produced. This means that marginal profit = marginal revenue – marginal cost.

[redacted]

**41.** True. The acceleration is the first derivative of the velocity which, in turn, is the second derivative of the position function.

[redacted]

**43.** D. $V(x) = x^3$

$$\frac{dv}{dx} = 3x^2$$

[redacted]

**45.** C. $v(t) = 7 - 2t = 0$

$$7 = 2t$$

$$t = \frac{7}{2}$$

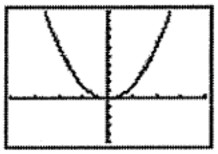
[redacted]

47. (a) $g'(x) = \frac{d}{dx}(x^3) = 3x^2$

$h'(x) = \frac{d}{dx}(x^3 - 2) = 3x^2$

$t'(x) = \frac{d}{dx}(x^3 + 3) = 3x^2$

(b) The graphs of NDER $g(x)$, NDER $h(x)$, and NDER $t(x)$ are all the same, as shown.



$[-4, 4]$ by $[-10, 20]$

(c) $f(x)$ must be of the form $f(x) = x^3 + c$, where $c$ is a constant.

(d) Yes. $f(x) = x^3$

(e) Yes. $f(x) = x^3 + 3$

[redacted]

49. (a) Assume that $f$ is even. Then,

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h},$$

and substituting $k = -h$,

$$= \lim_{k\to 0} \frac{f(x+k) - f(x)}{-k}$$
$$= -\lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x)$$

So, $f'$ is an odd function.

(b) Assume that $f$ is odd. Then,

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$$
$$= \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h},$$

and substituting $k = -h$,

$$= \lim_{k\to 0} \frac{-f(x+k) + f(x)}{-k}$$
$$= \lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x)$$

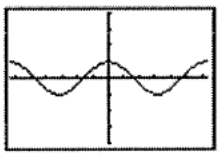
So, $f'$ is an even function.

[redacted]

## Section 3.5 Derivatives of Trigonometric Functions (pp. 141–147)

### Exploration 1 Making a Conjecture with NDER

1. When the graph of $\sin x$ is increasing, the graph of NDER ($\sin x$) is positive (above the $x$-axis).
2. When the graph of $\sin x$ is decreasing, the graph of NDER ($\sin x$) is negative (below the $x$-axis).
3. When the graph of $\sin x$ stops increasing and starts decreasing, the graph of NDER ($\sin x$) crosses the $x$-axis from above to below.
4. The slope of the graph of $\sin x$ matches the value of NDER ($\sin x$) at these points.
5. We conjecture that NDER ($\sin x$) = $\cos x$. The graphs coincide, supporting our conjecture.

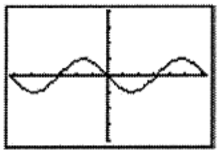


$[-2\pi, 2\pi]$ by $[-4, 4]$

6. When the graph of $\cos x$ is increasing, the graph of NDER ($\cos x$) is positive (above the $x$-axis).
When the graph of $\cos x$ is decreasing, the graph of NDER ($\cos x$) is negative (below the $x$-axis).
When the graph of $\cos x$ stops increasing and starts decreasing, the graph of NDER ($\cos x$) crosses the $x$-axis from above to below.
The slope of the graph of $\cos x$ matches the value of NDER ($\cos x$) at these points.

**6. Continued**

We conjecture that NDER (cos $x$) = −sin $x$. The graphs coincide, supporting our conjecture.



$[-2\pi, 2\pi]$ by $[-4, 4]$

## Quick Review 3.5

**1.** $135° \cdot \dfrac{\pi}{180°} = \dfrac{3\pi}{4} \approx 2.356$

**2.** $1.7 \cdot \dfrac{180°}{\pi} = \left(\dfrac{306}{\pi}\right)^{\circ} \approx 97.403°$

**3.** $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

**4.** Domain: All reals
Range: $[-1, 1]$

**5.** Domain: $x \neq \dfrac{k\pi}{2}$ for odd integers $k$
Range: All reals

**6.** $\cos a = \pm\sqrt{1-\sin^2 a} = \pm\sqrt{1-(-1)^2} = \pm\sqrt{0} = 0$

**7.** If $\tan a = -1$, then $a = \dfrac{3\pi}{4} + k\pi$ for some integer $k$,

so $\sin a = \pm\dfrac{1}{\sqrt{2}}$.

**8.** $\dfrac{1-\cos h}{h} = \dfrac{(1-\cos h)(1+\cos h)}{h(1+\cos h)} = \dfrac{1-\cos^2 h}{h(1+\cos h)}$

$= \dfrac{\sin^2 h}{h(1+\cos h)}$

**9.** $y'(x) = 6x^2 - 14x$
$y'(3) = 12$
The tangent line has slope 12 and passes through (3, 1), so its equation is $y = 12(x-3)+1$, or $y = 12x - 35$.

**10.** $a(t) = v'(t) = 6t^2 - 14t$
$a(3) = 12$

## Section 3.5 Exercises

**1.** $\dfrac{d}{dx}(1+x-\cos x) = 0 + 1 - (-\sin x) = 1 + \sin x$

[redacted]

**3.** $\dfrac{d}{dx}\left(\dfrac{1}{x} + 5\sin x\right) = -\dfrac{1}{x^2} + 5\cos x$

[redacted]

**5.** $\dfrac{d}{dx}(4 - x^2\sin x) = \dfrac{d}{dx}(4) - \left[x^2\dfrac{d}{dx}(\sin x) + (\sin x)\dfrac{d}{dx}(x^2)\right]$

$= 0 - [x^2\cos x + (\sin x)(2x)]$

$= -x^2\cos x - 2x\sin x$

[redacted]

**7.** $\dfrac{d}{dx}\left(\dfrac{4}{\cos x}\right) = \dfrac{d}{dx}(4\sec x) = 4\sec x\tan x$

[redacted]

**9.** $\dfrac{d}{dx}\dfrac{\cot x}{1+\cot x} = \dfrac{(1+\cot x)\dfrac{d}{dx}(\cot x) - (\cot x)\dfrac{d}{dx}(1+\cot x)}{(1+\cot x)^2}$

$= \dfrac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$

$= -\dfrac{\csc^2 x}{(1+\cot x)^2} = -\dfrac{\csc^2 x\sin^2 x}{(1+\cot x)^2\sin^2 x} = -\dfrac{1}{(\sin x + \cos x)^2}$

[redacted]

**11.** $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dx}(5\sin t)$
$v(t) = 5\cos t$
$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dx}(5\cos t)$
$a(t) = -5\sin t$

The weight starts at 0, goes to 5, and the oscillates between 5 and −5. The period of the motion is $2\pi$. The speed is greatest when $\cos t = \pm 1$ ($t = k\pi$), zero when

$\cos t = 0\ \left(t = \dfrac{k\pi}{2},\ k \text{ odd}\right)$. The acceleration is greatest

when $\sin t = \pm 1\ \left(t = \dfrac{k\pi}{2},\ k \text{ odd}\right)$, zero when

$\sin t = 0\ (t = k\pi)$.

[redacted]

**13. (a)** $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(2 + 3\sin t)$

$v(t) = 3\cos t$, speed $= |3\cos t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(3\cos t) = -3\sin t$

**(b)** $v\left(\dfrac{\pi}{4}\right) = 3\cos\left(\dfrac{\pi}{4}\right) = \dfrac{3\sqrt{2}}{2}$, speed $= \dfrac{3\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right) = -3\sin\left(\dfrac{\pi}{4}\right) = -\dfrac{3\sqrt{2}}{2}$

**(c)** The body starts at 2, goes up to 5, goes down to –1, and then oscillates between –1 and 5. The period of motion is $2\pi$.

[redacted]

**15. (a)** $v(t) = \dfrac{ds}{dt} = \dfrac{d}{dt}(2\sin t + 3\cos t)$

$v(t) = 2\cos t - 3\sin t$, speed $= |2\cos t - 3\sin t|$

$a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}(2\cos t - 3\sin t)$

$a(t) = -2\sin t - 3\cos t$

**(b)** $v\left(\dfrac{\pi}{4}\right) = 2\cos\dfrac{\pi}{4} - 3\sin\dfrac{\pi}{4}$

$v\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

speed $= \dfrac{\sqrt{2}}{2}$

$a\left(\dfrac{\pi}{4}\right) = -2\sin\dfrac{\pi}{4} - 3\cos\dfrac{\pi}{4}$

$a\left(\dfrac{\pi}{4}\right) = \dfrac{-5\sqrt{2}}{2}$

**(c)** The body starts at 3, goes to $3.606(\sqrt{13})$, and then oscillates between –3.606 and 3.606. The period of the motion is $2\pi$.

[redacted]

**17.** $j(t) = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

$f(t) = 2\cos t$

$f'(t) = -2\sin t$

$f''(t) = -2\cos t$

$f'''(t) = 2\sin t$

[redacted]

**19.** $j(t) = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$

$f(t) = \sin t - \cos t$

$f'(t) = \cos t + \sin t$

$f''(t) = -\sin t + \cos t$

$f'''(t) = -\cos t - \sin t$

[redacted]

**21.** $y = \sin x + 3$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + 3) = \cos x$$
$$y(\pi) = \sin \pi + 3 = 3$$
$$y'(\pi) = \cos \pi = -1$$

tangent: $y = -1(x - \pi) + 3 = -x + \pi + 3$

normal: $m_2 = -\frac{1}{m_1} = 1$

$$y = (x - \pi) + 3$$

[redacted]

**23.** $y = x^2 \sin x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x$$
$$y(3) = (3)^2 \sin 3 = 1.270$$
$$y'(3) = 2(3)\sin 3 + (3)^2 \cos 3 = -8.063$$

tangent: $y = -8.063(x - 3) + 1.270 = -8.063x + 25.460$

normal: $m_2 = -\frac{1}{m_1} = 0.124$

$$y = 0.124(x - 3) + 1.270$$
$$y = 0.124x + 0.898$$

[redacted]

**25. (a)** $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

**(b)** $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{(\cos x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$

$$= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

[redacted]

**27.** $\frac{d}{dx}\sec x = \sec x \tan x$ which is 0 at $x = 0$, so the slope of the tangent line is 0. $\frac{d}{dx}\cos x = -\sin x$ which is 0 at $x = 0$, so the slope of the tangent line is 0.

[redacted]

**29.** $y'(x)=\dfrac{d}{dx}(\sqrt{2}\cos x)=-\sqrt{2}\sin x$

$$y'\left(\frac{\pi}{4}\right)=-\sqrt{2}\sin\frac{\pi}{4}=-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=-1$$

The tangent line has slope $-1$ and passes through $\left(\frac{\pi}{4},\ \sqrt{2}\cos\frac{\pi}{4}\right)=\left(\frac{\pi}{4},\ 1\right)$, so its equation is

$y=-1\left(x-\frac{\pi}{4}\right)+1$, or $y=-x+\frac{\pi}{4}+1$.

The normal line has slope 1 and passes through $\left(\frac{\pi}{4},\ 1\right)$, so its equation is $y=1\left(x-\frac{\pi}{4}\right)+1$, or $y=x+1-\frac{\pi}{4}$.

[redacted]

**31.** $y'(x)=\dfrac{d}{dx}(4+\cot x-2\csc x)$

$$=0-\csc^2 x+2\csc x\cot x$$
$$=-\csc^2 x+2\csc x\cot x$$

**(a)** $y'\left(\frac{\pi}{2}\right)=-\csc^2\frac{\pi}{2}+2\csc\frac{\pi}{2}\cot\frac{\pi}{2}$

$$=-1^2+2(1)(0)=-1$$

The tangent line has slope $-1$ and passes through $P\left(\frac{\pi}{2},\ 2\right)$. Its equation is $y=-1\left(x-\frac{\pi}{2}\right)+2$, or

$y=-x+\frac{\pi}{2}+2.$

**(b)**
$$f'(x)=0$$
$$-\csc^2 x+2\csc x\cot x=0$$
$$-\frac{1}{\sin^2 x}+\frac{2\cos x}{\sin^2 x}=0$$
$$\frac{1}{\sin^2 x}(2\cos x-1)=0$$
$$\cos x=\frac{1}{2}$$
$$x=\frac{\pi}{3}\text{ at point }Q$$

$$y\left(\frac{\pi}{3}\right)=4+\cot\frac{\pi}{3}-2\csc\frac{\pi}{3}$$
$$=4+\frac{1}{\sqrt{3}}-2\left(\frac{2}{\sqrt{3}}\right)$$
$$=4-\frac{3}{\sqrt{3}}=4-\sqrt{3}$$

The coordinates of $Q$ are $\left(\frac{\pi}{3},\ 4-\sqrt{3}\right)$.

The equation of the horizontal line is $y=4-\sqrt{3}$.

[redacted]

**33. (a)** Velocity: $s'(t) = -2\cos t$ m/sec

Speed: $|s'(t)| = |2\cos t|$ m/sec

Acceleration: $s''(t) = 2\sin t \text{ m/sec}^2$

Jerk: $s'''(t) = 2\cos t \text{ m/sec}^3$

**(b)** Velocity: $-2\cos\frac{\pi}{4} = -\sqrt{2}$ m/sec

Speed: $|-\sqrt{2}| = \sqrt{2}$ m/sec

Acceleration: $2\sin\frac{\pi}{4} = \sqrt{2} \text{ m/sec}^2$

Jerk: $2\cos\frac{\pi}{4} = \sqrt{2} \text{ m/sec}^3$

**(c)** The body starts at 2, goes to 0 and then oscillates between 0 and 4.

Speed:

*Greatest* when $\cos t = \pm 1$ (or $t = k\pi$), at the center of the interval of motion.

*Zero* when $\cos t = 0$ (or $t = \frac{k\pi}{2}$, $k$ odd), at the endpoints of the interval of motion.

Acceleration:

*Greatest* (in magnitude) when $\sin t = \pm 1$ (or $t = \frac{k\pi}{2}$, $k$ odd)

*Zero* when $\sin t = 0$ (or $t = k\pi$)

Jerk:

*Greatest* (in magnitude) when $\cos t = \pm 1$ (or $t = k\pi$).

*Zero* when $\cos t = 0$ (or $t = \frac{k\pi}{2}$, $k$ odd)

[redacted]

[redacted]

*Greatest* (in magnitude) when $t = \frac{\pi}{4} + k\pi$

*Zero* when $t = \frac{3\pi}{4} + k\pi$

Jerk:

*Greatest* (in magnitude) when $t = \frac{3\pi}{4} + k\pi$

*Zero* when $t = \frac{\pi t}{4} + k\pi$

**35.** $y' = \frac{d}{dx}\csc x = -\csc x \cot x$

$$y'' = \frac{d}{dx}(-\csc x \cot x)$$
$$= -(\csc x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(\csc x)$$
$$= -(\csc x)(-\csc^2 x) - (\cot x)(-\csc x \cot x)$$
$$= \csc^3 x + \csc x \cot^2 x$$

[redacted]

**37.** Continuous:

Note that $g(0) = \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \cos x = \cos(0) = 1$, and $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(x+b) = b$. We require $\lim_{x\to 0^-} g(x) = g(0)$, so $b = 1$. The function is continuous if $b = 1$.

Differentiable:

For $b = 1$, the left-hand derivative is 1 and the right-hand derivative is $-\sin(0) = 0$, so the function is not differentiable. For other values of $b$, the function is discontinuous at $x = 0$ and there is no left-hand derivative. So, there is no value of $b$ that will make the function differentiable at $x = 0$.

[redacted]

**39.** Observe the pattern:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d^5}{dx^5}\sin x = \cos x$$
$$\frac{d^2}{dx^2}\sin x = -\sin x \qquad \frac{d^6}{dx^6}\sin x = -\sin x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x \qquad \frac{d^7}{dx^7}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\sin x = \sin x \qquad \frac{d^8}{dx^8}\sin x = \sin x$$

Continuing the pattern, we see that

$\frac{d^n}{dx^n}\sin x = \cos x$ when $n = 4k+1$ for any whole number $k$.

Since $725 = 4(181) + 1$, $\frac{d^{725}}{dx^{725}}\sin x = \cos x$.

[redacted]

**41.** **(a)** Using $y = x$, $\sin(0.12) \approx 0.12$.

**(b)** $\sin(0.12) \approx 0.1197122$; The approximation is within 0.0003 of the actual value.

[redacted]

**43.**
$$\begin{aligned}\frac{d}{dx}\cos 2x &= \frac{d}{dx}[(\cos x)(\cos x) - (\sin x)(\sin x)]\\ &= \left[(\cos x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(\cos x)\right] - \\ &\quad \left[(\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)\right]\\ &= 2(\cos x)(-\sin x) - 2(\sin x)(\cos x)\\ &= -4\sin x\cos x\\ &= -2(2\sin x\cos x)\\ &= -2\sin 2x\end{aligned}$$

[redacted]

**45.** False. The velocity is negative and the speed is positive at $t = \frac{\pi}{4}$.

[redacted]

**47.** B. See 46.

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$
$$y = (x - \pi) - 1$$

[redacted]

**49.** C. $v(t) = \frac{ds}{dt} = \frac{d}{dt}(3 + \sin t)$

$v(t) = \cos t = 0$

$t = \frac{\pi}{2}$

[redacted]

[redacted]

**51.** $\lim_{h\to 0}\frac{(\cos h-1)}{h}=\lim_{h\to 0}\frac{(\cos h-1)(\cos h+1)}{h(\cos h+1)}$

$$=\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}$$

$$=\lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)}$$

$$=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{\sin h}{\cos h+1}\right)$$

$$=-(1)\left(\frac{0}{2}\right)=0$$

[redacted]

## **Section 3.6** Chain Rule (pp. 148–156)

### Quick Review 3.6

**1.** $f(g(x)=f(x^2+1)=\sin(x^2+1)$

**2.** $f(g(h(x)))=f(g(7x))=f((7x)^2+1)$
$=\sin[(7x)^2+1]=\sin(49x^2+1)$

**3.** $(g\circ h)(x)=g(h(x))=g(7x)=(7x)^2+1=49x^2+1$

**4.** $(h\circ g)(x)=h(g(x))=h(x^2+1)=7(x^2+1)=7x^2+7$

**5.** $f\left(\frac{g(x)}{h(x)}\right)=f\left(\frac{x^2+1}{7x}\right)=\sin\frac{x^2+1}{7x}$

**6.** $\sqrt{\cos x+2}=g(\cos x)=g(f(x))$

**7.** $\sqrt{3\cos^2 x+2}=g(3\cos^2 x)=g(h(\cos x))=g(h(f(x)))$

**8.** $3\cos x+6=3(\cos x+2)=3(\sqrt{\cos x+2})^2$
$=h(\sqrt{\cos x+2})=h(g(\cos x))=h(g(f(x)))$

**9.** $\cos 27x^4=f(27x^4)=f(3(3x^2)^2)=f(h(3x^2))=f(h(h(x)))$

**10.** $\cos\sqrt{2+3x^2}=\cos\sqrt{3x^2+2}=f(\sqrt{3x^2+2})$
$=f(g(3x^2))=f(g(h(x)))$

## Section 3.6 Exercises

1. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = \sin u \qquad u = 3x + 1$

$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 3$

$\frac{dy}{dx} = 3\cos(3x+1)$

[redacted]

3. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = \cos u \qquad u = \sqrt{3}x$

$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = \sqrt{3}$

$\frac{dy}{dx} = -\sqrt{3}\sin(\sqrt{3}x)$

[redacted]

5. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = u^2 \qquad u = \frac{\sin x}{1+\cos x}$

$\frac{dy}{du} = 2u \qquad \frac{du}{dx} = \frac{\sin x}{(1+\cos x)^2}$

$\frac{dy}{dx} = \frac{2\sin x}{(1+\cos x)^2}$

[redacted]

7. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = \cos u \qquad u = \sin x$

$\frac{dy}{du} = -\sin u \qquad \frac{du}{dx} = \cos x$

$\frac{dy}{dx} = -\sin(\sin x)\cos x$

[redacted]

9. $\frac{ds}{dt} = \frac{d}{dt}\cos\left(\frac{\pi}{2} - 3t\right)$

$= \left[-\sin\left(\frac{\pi}{2} - 3t\right)\right]\frac{d}{dt}\left(\frac{\pi}{2} - 3t\right)$

$= \left[-\sin\left(\frac{\pi}{2} - 3t\right)\right](-3)$

$= 3\sin\left(\frac{\pi}{2} - 3t\right)$

[redacted]

11. $\frac{ds}{dt} = \frac{d}{dt}\left(\frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t\right)$

$= \frac{4}{3\pi}(\cos 3t)\frac{d}{dt}(3t) + \frac{4}{5\pi}(-\sin 5t)\frac{d}{dt}(5t)$

$= \frac{4}{3\pi}(\cos 3t)(3) + \frac{4}{5\pi}(-\sin 5t)(5)$

$= \frac{4}{\pi}\cos 3t - \frac{4}{\pi}\sin 5t$

[redacted]

**23.** $\frac{dy}{dx}=\frac{d}{dx}(1+\cos^2 7x)^3$

$=3(1+\cos^2 7x)^2\frac{d}{dx}(1+\cos^2 7x)$

$=3(1+\cos^2 7x)^2(2\cos 7x)\frac{d}{dx}(\cos 7x)$

$=3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)\frac{d}{dx}(7x)$

$=3(1+\cos^2 7x)^2(2\cos 7x)(-\sin 7x)(7)$

$=-42(1+\cos^2 7x)^2\cos 7x\sin 7x$

[redacted]

**25.** $\frac{dr}{d\theta}=\frac{d}{d\theta}\tan(2-\theta)=\sec^2(2-\theta)\frac{d}{d\theta}(2-\theta)$

$=\sec^2(2-\theta)(-1)=-\sec^2(2-\theta)$

[redacted]

**27.** $\frac{dr}{d\theta}=\frac{d}{d\theta}\sqrt{\theta\sin\theta}=\frac{1}{2\sqrt{\theta\sin\theta}}\frac{d}{d\theta}(\theta\sin\theta)$

$=\frac{1}{2\sqrt{\theta\sin\theta}}\left[\theta\frac{d}{d\theta}(\sin\theta)+(\sin\theta)\frac{d}{d\theta}(\theta)\right]$

$=\frac{1}{2\sqrt{\theta\sin\theta}}(\theta\cos\theta+\sin\theta)$

$=\frac{\theta\cos\theta+\sin\theta}{2\sqrt{\theta\sin\theta}}$

[redacted]

**29.** $y'=\frac{d}{dx}\tan x=\sec^2 x$

$y''=\frac{d}{dx}\sec^2 x=(2\sec x)\frac{d}{dx}(\sec x)$

$=(2\sec x)(\sec x\tan x)$

$=2\sec^2 x\tan x$

[redacted]

**31.** $y'=\frac{d}{dx}\cot(3x-1)=-\csc^2(3x-1)\frac{d}{dx}(3x-1)$

$=-3\csc^2(3x-1)$

$y''=\frac{d}{dx}[-3\csc^2(3x-1)]$

$=-3[2\csc(3x-1)]\frac{d}{dx}\csc(3x-1)$

$=-3[2\csc(3x-1)]\bullet$

$[-\csc(3x-1)\cot(3x-1)]\frac{d}{dx}(3x-1)$

$=-3[2\csc(3x-1)][-\csc(3x-1)\cot(3x-1)](3)$

$=18\csc^2(3x-1)\cot(3x-1)$

[redacted]

**33.** $f'(u)=\frac{d}{du}(u^5+1)=5u^4$

$g'(x)=\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$

$(f\circ g)'(1)=f'(g(1))g'(1)=f'(1)g'(1)=(5)\left(\frac{1}{2}\right)=\frac{5}{2}$

[redacted]

**35.** $f'(u) = \frac{d}{du}\left(\cot\frac{\pi u}{10}\right) = -\csc^2\left(\frac{\pi u}{10}\right)\frac{d}{du}\left(\frac{\pi u}{10}\right)$

$= -\frac{\pi}{10}\csc^2\left(\frac{\pi u}{10}\right)$

$g'(x) = \frac{d}{dx}(5\sqrt{x}) = \frac{5}{2\sqrt{x}}$

$(f \circ g)'(1) = f'(g(1))g'(1) = f'(5)g'(1)$

$= -\frac{\pi}{10}\left[\csc^2\left(\frac{\pi}{2}\right)\right]\left(\frac{5}{2}\right)$

$= -\frac{\pi}{10}(1)\left(\frac{5}{2}\right) = -\frac{\pi}{4}$

[redacted]

**37.** $f'(u) = \frac{d}{du}\frac{2u}{u^2+1} = \frac{(u^2+1)\frac{d}{du}(2u) - (2u)\frac{d}{du}(u^2+1)}{(u^2+1)^2}$

$= \frac{(u^2+1)(2) - (2u)(2u)}{(u^2+1)^2} = \frac{-2u^2+2}{(u^2+1)^2}$

$g'(x) = \frac{d}{dx}(10x^2 + x + 1) = 20x + 1$

$(f \circ g)'(0) = f'(g(0))g'(0) = f'(1)g'(0) = (0)(1) = 0$

[redacted]

**39. (a)** $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{d}{du}(\cos u)\frac{d}{dx}(6x+2)$

$= (-\sin u)(6)$

$= -6\sin u$

$= -6\sin(6x+2)$

**(b)** $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{d}{du}(\cos 2u)\frac{d}{dx}(3x+1)$

$= (-\sin 2u)(2) \cdot (3)$

$= -6\sin 2u$

$= -6\sin(6x+2)$

[redacted]

**41.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(2\cos t) = -2\sin t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}(2\sin t) = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-2\sin t} = -\cot t$$

This line passes through $\left(2\cos\dfrac{\pi}{4},\ 2\sin\dfrac{\pi}{4}\right) = (\sqrt{2},\ \sqrt{2})$

and has slope $-\cot\dfrac{\pi}{4} = -1$. Its equation is

$y = -(x - \sqrt{2}) + \sqrt{2}$, or $y = -x + 2\sqrt{2}$.

[redacted]

**43.** $\dfrac{dx}{dt} = \dfrac{d}{dt}(\sec^2 t - 1) = (2\sec t)\dfrac{d}{dt}(\sec t)$

$= (2\sec t)(\sec t \tan t)$

$= 2\sec^2 t \tan t$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\tan t = \sec^2 t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{2\sec^2 t \tan t} = \frac{1}{2}\cot t.$$

The line passes through

$\left(\sec^2\left(-\dfrac{\pi}{4}\right) - 1,\ \tan\left(-\dfrac{\pi}{4}\right)\right) = (1,\ -1)$ and has

slope $\dfrac{1}{2}\cot\left(-\dfrac{\pi}{4}\right) = -\dfrac{1}{2}$. Its equation

is $y = -\dfrac{1}{2}(x - 1) - 1$, or $y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

[redacted]

**45.** $\dfrac{dx}{dt} = \dfrac{d}{dt}t = 1$

$\dfrac{dy}{dt} = \dfrac{d}{dt}\sqrt{t} = \dfrac{1}{2\sqrt{t}}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1/(2\sqrt{t})}{1} = \frac{1}{2\sqrt{t}}$$

The line passes through $\left(\dfrac{1}{4}, \sqrt{\dfrac{1}{4}}\right) = \left(\dfrac{1}{4}, \dfrac{1}{2}\right)$ and has slope

$\dfrac{1}{2\sqrt{\frac{1}{4}}} = 1$. Its equation is $y = 1\left(x - \dfrac{1}{4}\right) + \dfrac{1}{2}$, or $y = x + \dfrac{1}{4}$.

[redacted]

**47.** $\frac{dx}{dt} = \frac{d}{dt}(t - \sin t) = 1 - \cos t$

$\frac{dy}{dt} = \frac{d}{dt}(1 - \cos t) = \sin t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$

The line passes through

$\left(\frac{\pi}{3} - \sin\frac{\pi}{3}, 1 - \cos\frac{\pi}{3}\right) = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and has slope

$\frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$. Its equation is

$y = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) + \frac{1}{2}$, or

$y = \sqrt{3}x + 2 - \frac{\pi}{\sqrt{3}}$.

[redacted]

**49. (a)** $\frac{dx}{dt} = \frac{d}{dt}(t^2 + t) = 2t + 1$

$\frac{dy}{dt} = \frac{d}{dt}\sin t = \cos t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{2t + 1}$

**(b)** $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\frac{\cos t}{2t + 1}$

$$= \frac{(2t+1)\frac{d}{dt}(\cos t) - (\cos t)\frac{d}{dt}(2t+1)}{(2t+1)^2}$$

$$= \frac{(2t+1)(-\sin t) - (\cos t)(2)}{(2t+1)^2}$$

$$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^2}$$

**(c)** Let $u = \frac{dy}{dx}$.

Then $\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt}$, so $\frac{du}{dx} = \frac{du}{dt} \div \frac{dx}{dt}$. Therefore,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$$

$$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^2} \div (2t+1)$$

$$= -\frac{(2t+1)(\sin t) + 2\cos t}{(2t+1)^3}$$

**(d)** The expression in part (c).

[redacted]

**51.** $\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt}$

$= (-\sin\theta)\left(\frac{d\theta}{dt}\right)$

When $\theta = \frac{3\pi}{2}$ and $\frac{d\theta}{dt} = 5$, $\frac{ds}{dt} = \left(-\sin\frac{3\pi}{2}\right)(5) = 5$.

[redacted]

**53.** $\frac{dy}{dx} = \frac{d}{dx}\sin\frac{x}{2} = \left(\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\cos\frac{x}{2}$

Since the range of the function $f(x) = \frac{1}{2}\cos\frac{x}{2}$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$,

the largest possible value of $\frac{dy}{dx}$ is $\frac{1}{2}$.

[redacted]

**55.** $\frac{dy}{dx} = \frac{d}{dx} 2\tan\frac{\pi x}{4} = \left(2\sec^2\frac{\pi x}{4}\right)\frac{d}{dx}\left(\frac{\pi x}{4}\right)$

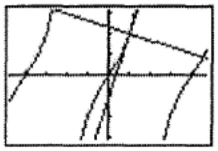
$= \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$

$y'(1) = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}(\sqrt{2})^2 = \pi.$

The tangent line has slope $\pi$ and passes through $\left(1, 2\tan\frac{\pi}{4}\right) = (1, 2)$. Its equation is $y = \pi(x-1)+2$, or $y = \pi x - \pi + 2$.

The normal line has slope $-\frac{1}{\pi}$ and passes through (1, 2). Its equation is $y = -\frac{1}{\pi}(x-1)+2$, or $y = -\frac{1}{\pi}x + \frac{1}{\pi} + 2$.

Graphical support:



[−4.7, 4.7] by [−3.1, 3.1]

[redacted]

[redacted]

**57.** $\frac{d}{dx}\cos(x°) = \frac{d}{dx}\cos\left(\frac{\pi x}{180}\right) = -\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right) = -\frac{\pi}{180}\sin(x°)$

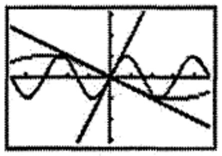
[redacted]

[redacted]

**59.** For $y = \sin 2x$, $y' = (\cos 2x)\frac{d}{dx}(2x) = 2\cos 2x$ and the slope at the origin is 2.

For $y = -\sin\frac{x}{2}$, $y' = \left(-\cos\frac{x}{2}\right)\frac{d}{dx}\left(\frac{x}{2}\right) = -\frac{1}{2}\cos\frac{x}{2}$ and the slope at the origin is $-\frac{1}{2}$. Since the slopes of the two tangent lines are 2 and $-\frac{1}{2}$, the lines are perpendicular and the curves are orthogonal.

A graph of the two curves along with the tangents $y = 2x$ and $y = -\frac{1}{2}x$ is shown.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

[redacted]

**61.** Velocity: $s'(t) = -2\pi bA\sin(2\pi bt)$
acceleration: $s''(t) = -4\pi^2 b^2 A\cos(2\pi bt)$
jerk: $s'''(t) = 8\pi^3 b^3 A\sin(2\pi bt)$
The velocity, amplitude, and jerk and proportional to $b$, $b^2$, and $b^3$ respectively. If the frequency $b$ is doubled, then the amplitude of the velocity is doubled, the amplitude of the acceleration is quadrupled, and the amplitude of the jerk is multiplied by 8.

[redacted]

**63.** Velocity: $s'(t) = \frac{d}{dt}\sqrt{1+4t} = \frac{1}{2\sqrt{1+4t}}\frac{d}{dt}(1+4t)$

$$= \frac{4}{2\sqrt{1+4t}} = \frac{2}{\sqrt{1+4t}}$$

At $t = 6$, the velocity is $\frac{2}{\sqrt{1+4(6)}} = \frac{2}{5}$ m/sec

**63. Continued**

Acceleration: $s''(t) = \frac{d}{dt}\frac{2}{\sqrt{1+4t}}$

$$= \frac{(\sqrt{1+4t})\frac{d}{dt}(2) - 2\frac{d}{dt}\sqrt{1+4t}}{(\sqrt{1+4t})^2}$$

$$= \frac{-2\left(\frac{1}{2\sqrt{1+4t}}\right)\frac{d}{dt}(1+4t)}{1+4t}$$

$$= \frac{\frac{-4}{\sqrt{1+4t}}}{1+4t} = -\frac{4}{(1+4t)^{3/2}}$$

At $t = 6$, the acceleration is $-\frac{4}{[1+4(6)]^{3/2}} = -\frac{4}{125}$ m/sec$^2$

[redacted]

**65.** Note that this Exercise concerns itself with the slowing down caused by the earth's atmosphere, *not* the acceleration caused by gravity.

Given: $v = \frac{k}{\sqrt{s}}$

Acceleration $= \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \left(\frac{dv}{ds}\right)(v) = (v)\left(\frac{dv}{ds}\right)$

$$= \left(\frac{k}{\sqrt{s}}\right)\frac{d}{ds}\frac{k}{\sqrt{s}}$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\sqrt{s}\frac{d}{ds}(k) - k\frac{d}{ds}\sqrt{s}}{(\sqrt{s})^2}\right)$$

$$= \left(\frac{k}{\sqrt{s}}\right)\left(\frac{\frac{-k}{(2\sqrt{s})}}{s}\right)$$

$$= -\frac{k^2}{2s^2}, s \geq 0$$

Thus, the acceleration is inversely proportional to $s^2$.

[redacted]

**67.** $\frac{dT}{du} = \frac{dT}{dL}\frac{dL}{du} = \left(\frac{d}{dL}2\pi\sqrt{\frac{L}{g}}\right)(kL)$

$$= \left(2\pi\frac{1}{2\sqrt{\frac{L}{g}}}\right)\left(\frac{d}{dL}\frac{L}{g}\right)(kL)$$

$$= \left(\frac{\pi}{\sqrt{\frac{L}{g}}}\right)\left(\frac{1}{g}\right)(kL) = k\pi\sqrt{\frac{L}{g}} = \frac{kT}{2}$$

[redacted]

**69.** Yes. Note that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, then $f'(g(1))g'(1) = 0$, so either $g'(1) = 0$ or $f'(g(1)) = 0$. This means that either the graph of $y = g(x)$ has a horizontal tangent at $x = 1$, or the graph of $y = f(u)$ has a horizontal tangent at $u = g(1)$.

[redacted]

**71.** False. It is +1.

[redacted]

**73.** C. $\frac{dy}{dx} = \frac{d}{dx}\cos^2(x^3 + x^2)$

$y = \cos^2 u \qquad u = x^3 + x^2$

$\frac{dy}{du} = -2\sin u\cos u \qquad \frac{du}{dx} = 3x^2 + 2x$

$$\frac{dy}{dx} = -2(3x^2 + 2x)\cos(3x^2 + 2x)\sin(3x^2 + 2x)$$

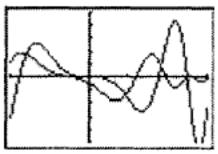
[redacted]

**75.** B. See problem 74.

$$\frac{dy}{dx} = \frac{\cos t}{1 + \sin t} = 0$$
$$\cos t = 0$$
$$t = \frac{\pi}{2}$$

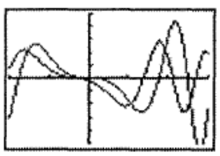
[redacted]

**77.** For $h = 1$:



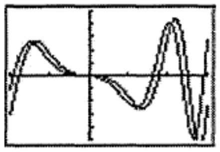
$[-2, 3]$ by $[-5, 5]$

For $h = 0.3$:



$[-2, 3]$ by $[-5, 5]$

For $h = 0.3$:



$[-2, 3]$ by $[-5, 5]$

As $h \to 0$, the second curve (the difference quotient) approaches the first ($y = -2x\sin(x^2)$). This is because $-2x\sin(x^2)$ is the derivative of $\cos(x^2)$, and the second curve is the difference quotient used to define the derivative of $\cos(x^2)$. As $h \to 0$, the difference quotient expression should be approaching the derivative.

[redacted]

**79.** $$\frac{dG}{dx} = \frac{d}{dx}\sqrt{uv} = \frac{d}{dx}\sqrt{x(x+c)} = \frac{d}{dx}\sqrt{x^2+cx}$$
$$= \frac{1}{2\sqrt{x^2+cx}}\frac{d}{dx}(x^2+cx) = \frac{2x+c}{2\sqrt{x^2+cx}} = \frac{x+(x+c)}{2\sqrt{x(x+c)}}$$
$$= \frac{u+v}{2\sqrt{uv}} = \frac{A}{G}$$

## Quick Quiz Sections 3.4-3.6

**1.** B. $y=\sin^4 u \qquad u=3x$

$$\frac{dy}{du}=4\sin^3 u\cos u \qquad \frac{du}{dx}=3$$

$$\frac{dy}{dx}=12\sin^3(3x)\cos(3x)$$

**2.** A. $y=\cos x+\tan x$

$$y'=-\sin x+\sec^2 x$$

$$y''=-\cos x+2\sec^2 x\tan x$$

**3.** C. $x=3\sin t \qquad y=2\cos t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt}=\frac{d}{dt}(2\cos t)=-2\sin t$$

$$\frac{dx}{dt}=\frac{d}{dt}(3\sin t)=3\cos t$$

$$\frac{dy}{dx}=\frac{-2\sin t}{3\cos t}=-\frac{2}{3}\tan t.$$

**4.** **(a)** $s(0)=-(0)^2+(0)+2$

$s(0)=2$

$(0, 2)$

**(b)** $-t^2+t+2>0$

$(t+1)(-t+2)>0$

$-1<t<\frac{1}{2}$ but $t<0$ not real

$0\le t<\frac{1}{2}$

**(c)** $t>\frac{1}{2}$

**(d)** $v(t)=\frac{ds}{dt}=\frac{d}{dt}(-t^2+t+2)$

$=-2t+1$

**(e)** $a(t)=\frac{dv}{dt}=\frac{d}{dt}(-2t+1)$

$=-2$

**(f)** $t=\frac{1}{2}$

## Section 3.7 Implicit Differentiation (pp. 157–164)

### Exploration 1 An Unexpected Derivative

**1.** $2x-2y-2xy'+2yy'=0$. Solving for $y'$, we find that

$\frac{dy}{dx}=1$ (provided $y\neq x$).

**2.** With a constant derivative of 1, the graph would seem to be a line with slope 1.

**3.** Letting $x=0$ in the original equation, we find that $y=\pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y=x+2$ and $y=x-2$.

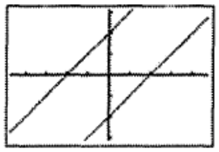
**4.** Factoring the original equation, we have

$[(x-y)-2][(x-y)+2]=0$

$\therefore x-y-2=0$ or $x-y+2=0$

$\therefore y=x-2$ or $y=x+2$.

The graph is shown below.



[–4.7, 4.7] by [–3.1, 3.1]

**5.** At each point $(x, y)$ on either line, $\frac{dy}{dx}=1$. The condition $y\neq x$ is true because both lines are parallel to the line $y=x$. The derivative is surprising because it does not depend on $x$ or $y$, but there are no inconsistencies.

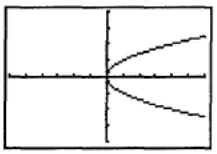
### Quick Review 3.7

**1.** $x-y^2=0$

$x=y^2$

$\pm\sqrt{x}=y$

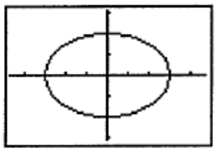
$y_1=\sqrt{x},\ y_2=-\sqrt{x}$



[-6, 6] by [-4, 4]

**2.** $4x^2 + 9y^2 = 36$

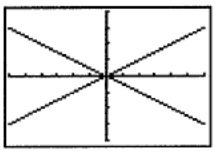
$$9y^2 = 36 - 4x^2$$
$$y^2 = \frac{36-4x^2}{9} = \frac{4}{9}(9 - x^2)$$
$$y = \pm\frac{2}{3}\sqrt{9-x^2}$$
$$y_1 = \frac{2}{3}\sqrt{9-x^2},\ y_2 = -\frac{2}{3}\sqrt{9-x^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

**3.** $x^2 - 4y^2 = 0$

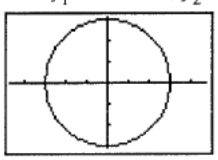
$$(x+2y)(x-2y) = 0$$
$$y = \pm\frac{x}{2}$$
$$y_1 = \frac{x}{2},\ y_2 = -\frac{x}{2}$$



[-6, 6] by [-4, 4]

**4.** $x^2 + y^2 = 9$

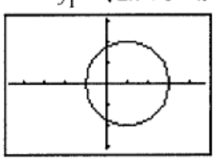
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9-x^2}$$
$$y_1 = \sqrt{9-x^2},\ y_2 = -\sqrt{9-y^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

**5.** $x^2 + y^2 = 2x + 3$

$$y^2 = 2x + 3 - x^2$$
$$y = \pm\sqrt{2x+3-x^2}$$
$$y_1 = \sqrt{2x+3-x^2},\ y^2 = -\sqrt{2x+3-x^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

**6.** $x^2y' - 2xy = 4x - y$

$$x^2y' = 4x - y + 2xy$$
$$y' = \frac{4x - y + 2xy}{x^2}$$

**7.** $y'\sin x - x\cos x = xy' + y$

$$y'\sin x - xy' = y + x\cos x$$
$$(\sin x - x)y' = y + x\cos x$$
$$y' = \frac{y + x\cos x}{\sin x - x}$$

**8.** $x(y^2 - y') = y'(x^2 - y)$

$$xy^2 = y'(x^2 - y + x)$$
$$y' = \frac{xy^2}{x^2 - y + x}$$

**9.** $\sqrt{x}(x - \sqrt[3]{x} = x^{1/2}(x - x^{1/3})$

$$= x^{1/2}x - x^{1/2}x^{1/3}$$
$$= x^{3/2} - x^{5/6}$$

**10.** $\frac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} = \frac{x + x^{2/3}}{x^{3/2}}$

$$= \frac{x}{x^{3/2}} + \frac{x^{2/3}}{x^{3/2}}$$
$$= x^{-1/2} + x^{-5/6}$$

## Section 3.7 Exercises

**1.** $x^2y + xy^2 = 6$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) = \frac{d}{dx}(6)$$
$$x^2\frac{dy}{dx} + y(2x) + x(2y)\frac{dy}{dx} + y^2(1) = 0$$
$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -(2xy + y^2)$$
$$(2xy + x^2)\frac{dy}{dx} = -(2xy + y^2)$$
$$\frac{dy}{dx} = -\frac{2xy + y^2}{2xy + x^2}$$

[redacted]

**3.** $y^2 = \dfrac{x-1}{x+1}$

$$\frac{d}{dx}y^2 = \frac{d}{dx}\frac{x-1}{x+1}$$

$$2y\frac{dy}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$

$$2y\frac{dy}{dx} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

[redacted]

**5.** $x = \tan y$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = \sec^2 y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

[redacted]

**7.** $x + \tan xy = 0$

$$\frac{d}{dx}(x) + \frac{d}{dx}(\tan xy) = \frac{d}{dx}(0)$$

$$1 + \sec^2(xy)\frac{d}{dx}(xy) = 0$$

$$1 + (\sec^2 xy)[x\frac{dy}{dx} + (y)\ (1)] = 0$$

$$(\sec^2 xy)\ (x)\frac{dy}{dx} = -1 - (\sec^2 xy)(y)$$

$$\frac{dy}{dx} = \frac{-1 - y\sec^2 xy}{x\sec^2 xy}$$

$$\frac{dy}{dx} = -\frac{1}{x}\cos^2 xy - \frac{y}{x}$$

[redacted]

**9.** $\dfrac{d}{dx}(x^2 + y^2) = \dfrac{d}{dx}(13)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y},\ -\frac{-2}{3} = \frac{2}{3}$$

[redacted]

**11.** $\frac{d}{dx}((x-1)^2+(y-1)^2)=\frac{d}{dx}(13)$

$$2(x-1)1+(2(y-1)1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x-1}{y-1},\ -\frac{3-1}{4-1}=-\frac{2}{3}$$

[redacted]

**13.** $\frac{d}{dx}(x^2y-xy^2)=\frac{d}{dx}(4)$

$$2xy-y^2+(2xy-x^2)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2xy-y^2}{2xy-x^2},$$

defined at every point except where $x=0$ or $y=\frac{x}{2}$.

[redacted]

**15.** $\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(xy)$

$$3x^2+3y^2\frac{dy}{dx}=y+x\frac{dx}{dy}$$

$$3x^2-y=(x-3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{3x^2-y}{x-3y^2},$$

defined everywhere except where $y^2=\frac{x}{3}$

[redacted]

**17.** $x^2+xy-y^2=1$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(xy)-\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$2x+x\frac{dy}{dx}+(y)(1)-2y\frac{dy}{dx}=0$$

$$(x-2y)\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}=\frac{-2x-y}{x-2y}=\frac{2x+y}{2y-x}$$

Slope at (2, 3): $\frac{2(2)+3}{2(1)-2}=\frac{7}{4}$

**(a)** Tangent: $y=\frac{7}{4}(x-2)+3$ or $y=\frac{7}{4}x-\frac{1}{2}$

**(b)** Normal: $y=-\frac{4}{7}(x-2)+3$ or $y=-\frac{4}{7}x+\frac{29}{7}$

[redacted]

**19.** $x^2y^2=9$

$$\frac{d}{dx}(x^2y^2)=\frac{d}{dx}(9)$$

$$(x^2)(2y)\frac{dy}{dx}+(y^2)(2x)=0$$

$$2x^2y\frac{dy}{dx}=-2xy^2$$

$$\frac{dy}{dx}=-\frac{2xy^2}{2x^2y}=-\frac{y}{x}$$

Slope at (−1, 3): $-\frac{3}{-1}=3$

**(a)** Tangent: $y=3(x+1)+3$ or $y=3x+6$

**(b)** Normal: $y=-\frac{1}{3}(x+1)+3$ or $y=-\frac{1}{3}x+\frac{8}{3}$

[redacted]

**21.** $$6x^2+3xy+2y^2+17y-6=0$$

$$\frac{d}{dx}(6x^2)+\frac{d}{dx}(3xy)+\frac{d}{dx}(2y^2)+\frac{d}{dx}(17y)-\frac{d}{dx}(6)=\frac{d}{dx}(0)$$

$$12x+3x\frac{dy}{dx}+(3y)(1)+4y\frac{dy}{dx}+17\frac{dy}{dx}-0=0$$

$$3x\frac{dy}{dx}+4y\frac{dy}{dx}+17\frac{dy}{dx}=-12x-3y$$

$$(3x+4y+17)\frac{dy}{dx}=-12x-3y$$

$$\frac{dy}{dx}=\frac{-12x-3y}{3x+4y+17}$$

Slope at $(-1, 0)$: $\frac{-12(-1)-3(0)}{3(-1)+4(0)+17}=\frac{12}{14}=\frac{6}{7}$

**(a)** Tangent: $y=\frac{6}{7}(x+1)+0$ or $y=\frac{6}{7}x+\frac{6}{7}$

**(b)** Normal: $y=-\frac{7}{6}(x+1)+0$ or $y=-\frac{7}{6}x-\frac{7}{6}$

[redacted]

**23.** $$2xy+\pi\sin y=2\pi$$

$$2\frac{d}{dx}(xy)+\pi\frac{d}{dx}(\sin y)=\frac{d}{dx}(2\pi)$$

$$2x\frac{dy}{dx}+2y(1)+\pi\cos y\frac{dy}{dx}=0$$

$$(2x+\pi\cos y)\frac{dy}{dx}=-2y$$

$$\frac{dy}{dx}=-\frac{2y}{2x+\pi\cos y}$$

Slope at $\left(1,\frac{\pi}{2}\right)$: $-\frac{2(\pi/2)}{2(1)+\pi\cos(\pi/2)}=-\frac{\pi}{2}$

**(a)** Tangent: $y=-\frac{\pi}{2}(x-1)+\frac{\pi}{2}$ or $y=-\frac{\pi}{2}x+\pi$

**(b)** Normal: $y=\frac{2}{\pi}(x-1)+\frac{\pi}{2}$ or $y=\frac{2}{\pi}x-\frac{2}{\pi}+\frac{\pi}{2}$

[redacted]

**25.** $$y=2\sin(\pi x-y)$$

$$\frac{dy}{dx}=\frac{d}{dx}2\sin(\pi x-y)$$

$$\frac{dy}{dx}=2\cos(\pi x-y)\left(\pi-\frac{dy}{dx}\right)$$

$$[1+2\cos(\pi x-y)]\frac{dy}{dx}=2\pi\cos(\pi x-y)$$

$$\frac{dy}{dx}=\frac{2\pi\cos(\pi x-y)}{1+2\cos(\pi x-y)}$$

Slope at $(1, 0)$: $\frac{2\pi\cos\pi}{1+2\cos\pi}=\frac{2\pi(-1)}{1+2(-1)}=2\pi$

**(a)** Tangent: $y=2\pi(x-1)+0$ or $y=2\pi x-2\pi$

**(b)** Normal: $y=-\frac{1}{2\pi}(x-1)+0$ or $y=-\frac{x}{2\pi}+\frac{1}{2\pi}$

[redacted]

**27.** $x^2 + y^2 = 1$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$
$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = -\frac{x}{y}$$
$$y'' = \frac{d}{dx}\left(-\frac{x}{y}\right)$$
$$= -\frac{(y)(1) - (x)(y')}{y^2}$$
$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$
$$= -\frac{x^2 + y^2}{y^3}$$

Since our original equation was $x^2 + y^2 = 1$, we may substitute 1 for $x^2 + y^2$, giving $y'' = -\frac{1}{y^3}$.

[redacted]

[redacted]

**29.** $y^2 = x^2 + 2x$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x)$$
$$2yy' = 2x + 2$$
$$y' = \frac{2x+2}{2y} = \frac{x+1}{y}$$
$$y'' = \frac{d}{dx}\left(\frac{x+1}{y}\right)$$
$$= \frac{(y)(1) - (x+1)y'}{y^2}$$
$$= \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$$
$$= \frac{y^2 - (x+1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may write $y^2 - (x+1)^2 = (x^2 + 2x) - (x^2 + 2x + 1) = -1$, which gives $y = -\frac{1}{y^3}$.

[redacted]

[redacted]

**31.** $\frac{dy}{dx}=\frac{d}{dx}x^{9/4}=\frac{9}{4}x^{(9/4)-1}=\frac{9}{4}x^{5/4}$

[redacted]

**33.** $\frac{dy}{dx}=\frac{d}{dx}\sqrt[3]{x}=\frac{d}{dx}x^{1/3}=\frac{1}{3}x^{(1/3)-1}=\frac{1}{3}x^{-2/3}$

[redacted]

**35.** $\frac{dy}{dx}=\frac{d}{dx}(2x+5)^{-1/2}=-\frac{1}{2}(2x+5)^{(-1/2)-1}\frac{d}{dx}(2x+5)$

$=-\frac{1}{2}(2x+5)^{-3/2}(2)=-(2x+5)^{-3/2}$

[redacted]

**37.** $\frac{dy}{dx}=\frac{d}{dx}\left(x\sqrt{x^2+1}\right)$

$=x\frac{d}{dx}\sqrt{x^2+1}+\sqrt{x^2+1}\frac{d}{dx}(x)$

$=x\frac{d}{dx}(x^2+1)^{1/2}+(x^2+1)^{1/2}$

$=x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)+(x^2+1)^{1/2}$

$=x^2(x^2+1)^{-1/2}+(x^2+1)^{1/2}$

Note: This answer is equivalent to $\frac{2x^2+1}{\sqrt{x^2+1}}$.

[redacted]

**39.** $\frac{dy}{dx}=\frac{d}{dx}(1-x^{1/2})^{1/2}$

$=\frac{1}{2}(1-x^{1/2})^{-1/2}\frac{d}{dx}(1-x^{1/2})$

$=\frac{1}{2}(1-x^{1/2})^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right)$

$=-\frac{1}{4}(1-x^{1/2})^{-1/2}x^{-1/2}$

[redacted]

**41.** $\frac{dy}{dx}=\frac{d}{dx}3(\csc x)^{3/2}$

$=\frac{9}{2}(\csc x)^{1/2}\frac{d}{dx}(\csc x)$

$=\frac{9}{2}(\csc x)^{1/2}(-\csc x\cot x)$

$=-\frac{9}{2}(\csc x)^{3/2}\cot x$

[redacted]

**43. (a)** If $f(x)=\frac{3}{2}x^{2/3}-3$, then

$f'(x)=x^{-1/3}$ and $f''(x)=-\frac{1}{3}x^{-4/3}$

which contradicts the given equation $f''(x)=x^{-1/3}$.

**(b)** If $f(x)=\frac{9}{10}x^{5/3}-7$, then

$f'(x)=\frac{3}{2}x^{2/3}$ and $f''(x)=x^{-1/3}$, which matches the given equation.

**(c)** Differentiating both sides of the given equation $f''(x)=x^{-1/3}$ gives $f'''(x)=-\frac{1}{3}x^{-4/3}$, so it *must* be true that $f'''(x)=-\frac{1}{3}x^{-4/3}$.

**(d)** If $f'(x)=\frac{3}{2}x^{2/3}+6$, then $f''(x)=x^{-1/3}$, which matches the given equation.

Conclusion: (b), (c), and (d) could be true.

[redacted]

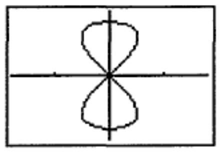
**45. (a)**
$$y^4 = y^2 - x^2$$
$$\frac{d}{dx}(y^4) = \frac{d}{dx}(y^2) - \frac{d}{dx}x^2$$
$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$
$$(4y^3 - 2y)\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y} = \frac{x}{y - 2y^3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$:

$$\text{Slope} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right)^3}$$
$$= \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{4}} \cdot \frac{\frac{4}{\sqrt{3}}}{\frac{4}{\sqrt{3}}} = \frac{1}{2-3} = -1$$

At $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$:

$$\text{Slope} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - 2\left(\frac{1}{2}\right)^3} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{1}{4}} \cdot \frac{4}{4} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

**(b)** 

[−1.8, 1.8] by [−1.2, 1.2]
Parameter interval: $-1 \le t \le 1$

[redacted]

**47. (a)** $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal $-1$.

**(b)**
$$x^3y^2 = \cos(\pi y)$$
$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}\cos(\pi y)$$
$$(x^3)(2y)\frac{dy}{dx} + (y^2)(3x^2) = (-\sin \pi y)(\pi)\frac{dy}{dx}$$
$$(2x^3y + \pi \sin \pi y)\frac{dy}{dx} = -3x^2y^2$$
$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \pi \sin \pi y}$$

Slope at $(-1, 1): -\frac{3(-1)^2(1)}{2(-1)^3(1) + \pi \sin \pi} = \frac{-3}{-2} = \frac{3}{2}$

The slope of the tangent line is $\frac{3}{2}$.

[redacted]

[redacted]

**49.** Find the two points:

The curve crosses the $x$-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$.

Show tangents are parallel:

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Slope at $(\sqrt{7}, 0): -\frac{2\sqrt{7}) + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0): -\frac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is –2.

[redacted]

**51.** First curve:

$$2x^2 + 3y^2 = 5$$

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(5)$$

$$4x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x}{6y} = -\frac{2x}{3y}$$

**51. Continued**

Second curve:

$$y^2 = x^3$$
$$\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$
$$2y\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At (1, 1), the slopes are $-\frac{2}{3}$ and $\frac{3}{2}$ respectively.

At (1, −1), the slopes are $\frac{2}{3}$ and $-\frac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\frac{5}{2}}\cos t,\ y = \sqrt{\frac{5}{3}}\sin t$

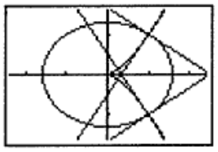
Second curve: $x = \sqrt[3]{t^2},\ y = t$

Tangents at (1, 1): $x = 1 + 3t, y = 1 - 2t$

$x = 1 + 2t,\ y = 1 + 3t$

Tangents at (1, −1): $x = 1 + 3t, y = -1 + 2t$

$x = 1 + 2t,\ y = -1 - 3t$



[−2.4, 2.4] by [−1.6, 1.6]

[redacted]

**53.** Acceleration $= \frac{dv}{dt} = \frac{d}{dt}[8(s-t)^{1/2} + 1]$

$$= 4(s-t)^{-1/2}\left(\frac{ds}{dt} - 1\right)$$
$$= 4(s-t)^{-1/2}(v-1)$$
$$= 4(s-t)^{-1/2}[(8(s-t)^{1/2} + 1) - 1]$$
$$= 32(s-t)^{-1/2}(s-t)^{1/2}$$
$$= 32 \text{ ft/sec}^2$$

[redacted]

**55. (a)**

$$x^3 + y^3 - 9xy = 0$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0)$$
$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9(y)(1) = 0$$
$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$
$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Slope at (4, 2): $\frac{3(2) - (4)^2}{(2)^2 - 3(4)} = \frac{-10}{-8} = \frac{5}{4}$

Slope at (2, 4): $\frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$

**(b)** The tangent is horizontal when

$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} = 0$, or $y = \frac{x^2}{3}$.

Substituting $\frac{x^2}{3}$ for $y$ in the original equation, we have:

$$x^3 + y^3 - 9xy = 0$$
$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$
$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$
$$\frac{x^3}{27}(x^3 - 54) = 0$$
$$x = 0 \text{ or } x = \sqrt[3]{54} = 3\sqrt[3]{2}$$

At $x = 0$, we have $y = \frac{0^2}{3} = 0$, which gives the point (0, 0), which is the origin. At $x = 3\sqrt[3]{2}$, we have $y = \frac{1}{3}(3\sqrt[3]{2})^2 = \frac{1}{3}(9\sqrt[3]{4}) = 3\sqrt[3]{4}$, so the point other than the origin is $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ or approximately (3.780, 4.762).

**55. Continued**

(c) The equation $x^3 + y^3 - 9xy$ is not affected by interchanging $x$ and $y$, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the $x$-value and the $y$-value in the answer to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or approximately (4.762, 3.780).

[redacted]

**57.**
$$xy + 2x - y = 0$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$
$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$
$$(x-1)\frac{dy}{dx} = -2 - y$$
$$\frac{dy}{dx} = \frac{-2-y}{x-1} = \frac{2+y}{1-x}$$

Since the slope of the line $2x + y = 0$ is $-2$, we wish to find points where the normal has slope $-2$, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have

$$\frac{2+y}{1-x} = \frac{1}{2}$$
$$2(2+y) = 1 - x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y-3)y + 2(-2y-3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y+1)(y+3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1, x = -2y - 3 = 2 - 3 = -1$.
At $y = -3: x = -2y - 3 = 6 - 3 = 3$.

The desired points are $(-1, -1)$ and $(3, -3)$.
Finally, we find the desired normals to the curve, which are the lines of slope $-2$ passing through each of these points. At $(-1, -1)$, the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At $(3, -3)$, the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

[redacted]

**59.** False.
$$\frac{d}{dx}(xy^2 + x) = \frac{d}{dx}(1)$$
$$y^2 + 1 + 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-1-y}{2xy},\ \frac{-1-1}{2\left(\frac{1}{2}\right)(1)} = -2$$

[redacted]

**61.** A. $\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(1)$

$$2x - y + (-x + 2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

[redacted]

**63.** E. $\frac{d}{dx}(y) = \frac{d}{dx}x^{3/4}$

$$\frac{dy}{dx} = \frac{3}{4}x^{-1/4} = \frac{3}{4x^{1/4}}$$

[redacted]

**65.** **(a)**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{d}{dx}(b^2x^2) + \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2b^2x}{2a^2y} = \frac{b^2x}{a^2y}$$

The slope at $(x_1, y_1)$ is $-\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$. This gives:

$a^2y_1y - a^2y_1{}^2 = -b^2x_1x + b^2x_1{}^2$
$a^2y_1y + b^2x_1x = a^2y_1{}^2 + b^2x_1{}^2.$

But $a^2y_1{}^2 + b^2x_1{}^2 = a^2b^2$ since $(x_1, y_1)$ is on the ellipse.

Therefore, $a^2y_1y + b^2x_1x = a^2b^2$, and dividing by $a^2b^2$ gives $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

**(b)**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$\frac{d}{dx}(b^2x^2) - \frac{d}{dx}(a^2y^2) = \frac{d}{dx}(a^2b^2)$$

$$2b^2x - 2a^2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2b^2x}{-2a^2y} = \frac{b^2x}{a^2y}$$

The slope at $(x_1, y_1)$ is $\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$.

This gives:

$a^2y_1y - a^2y_1{}^2 = b^2x_1x - b^2x_1{}^2$
$b^2x_1{}^2 - a^2y_1{}^2 = b^2x_1x - a^2y_1y$

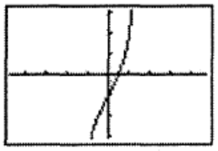
But $b^2x_1{}^2 - a^2y_1{}^2 = a^2b^2$ since $(x_1, y_1)$ is on the hyperbola. Therefore, $b^2x_1x - a^2y_1y = a^2b^2$, and dividing by $a^2b^2$ gives $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$.

[redacted]

## **Section 3.8** Derivatives of Inverse Trigonometric Functions (pp. 165–171)

### Exploration 1 Finding a derivative on an Inverse Graph Geometrically

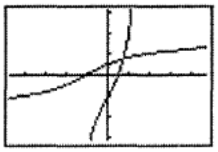
1. The graph is shown at the right. It appears to be a one-to-one function



[−4.7, 4.7] by [−3.1, 3.1]

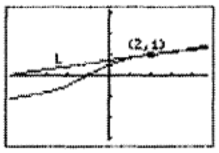
2. $f'(x) = 5x^4 + 2$. The fact that this function is always positive enables us to conclude that $f$ is everywhere increasing, and hence one-to-one.

3. The graph of $f^{-1}$ is shown to the right, along with the graph of $f$. The graph of $f^{-1}$ is obtained from the graph of $f$ by reflecting it in the line $y = x$.



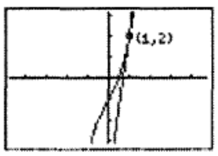
[−4.7, 4.7] by [−3.1, 3.1]

4. The line $L$ is tangent to the graph of $f^{-1}$ at the point $(2, 1)$.




[−4.7, 4.7] by [−3.1, 3.1]

5. The reflection of line $L$ is tangent to the graph of $f$ at the point




[−4.7, 4.7] by [−3.1, 3.1]

6. The reflection of the line $L$ is the tangent line to the graph of $y = x^5 + 2x - 1$ at the point $(1, 2)$. The slope is $\frac{dy}{dx}$ at $x = 1$, which is 7.

7. The slope of $L$ is the reciprocal of the slope of its reflection $\left(\text{since } \frac{\Delta y}{\Delta x} \text{ gets reflected to become } \frac{\Delta x}{\Delta y}\right)$. It is $\frac{1}{7}$.

8. $\frac{1}{7}$

### Quick Review 3.8

1. Domain: $[-1, 1]$

   Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   At 1: $\frac{\pi}{2}$

2. Domain: $[-1, 1]$

   Range: $[0, \pi]$

   At 1: 0

3. Domain: all reals

   Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   At 1: $\frac{\pi}{4}$

4. Domain: $(-\infty, -1] \cup [1, \infty)$

   Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

   At 1: 0

5. Domain: all reals

   Range: all reals

   At 1: 1

6. $f(x) = y = 3x - 8$

   $y + 8 = 3x$

   $x = \frac{y+8}{3}$

   Interchange $x$ and $y$:

   $y = \frac{x+8}{3}$

   $f^{-1}(x) = \frac{x+8}{3}$

7. $f(x) = y = \sqrt[3]{x+5}$

   $y^3 = x + 5$

   $x = y^3 - 5$

   Interchange $x$ and $y$:

   $y = x^3 - 5$

   $f^{-1}(x) = x^3 - 5$

8. $f(x) = y = \frac{8}{x}$

   $x = \frac{8}{y}$

   Interchange $x$ and $y$:

   $y = \frac{8}{x}$

   $f^{-1}(x) = \frac{8}{x}$

**9.** $f(x) = y = \frac{3x-2}{x}$

$$xy = 3x - 2$$
$$(y-3)x = -2$$
$$x = \frac{-2}{y-3} = \frac{2}{3-y}$$

Interchange $x$ and $y$:

$$y = \frac{2}{3-x}$$
$$f^{-1}(x) = \frac{2}{3-x}$$

**10.** $f(x) = y = \arctan\frac{x}{3}$

$$\tan y = \frac{x}{3}, -\frac{\pi}{2} < y < \frac{\pi}{2}$$
$$x = 3\tan y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Interchange $x$ and $y$:

$$y = 3\tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$f^{-1}(x) = 3\tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

## Section 3.8 Exercises

**1.** $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(x^2) = -\frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2)$

$$= -\frac{1}{\sqrt{1-x^4}}(2x) = -\frac{2x}{\sqrt{1-x^4}}$$

[redacted]

**3.** $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\sqrt{2}t = \frac{1}{\sqrt{1-(\sqrt{2}t)^2}}\frac{d}{dt}(\sqrt{2}t) = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

[redacted]

**5.** $\frac{dy}{dt} = \frac{d}{dt}\sin^{-1}\left(\frac{3}{t^2}\right) = \frac{1}{\sqrt{1-\left(\frac{3}{t^2}\right)^2}}\frac{d}{dt}\left(\frac{3}{t^2}\right)$

$$= \frac{1}{\sqrt{1-\frac{9}{t^4}}}\left(-\frac{6}{t^3}\right) = -\frac{6}{t\sqrt{t^4-9}}$$

[redacted]

**7.** $\frac{dy}{dx} = \frac{d}{dx}(x\sin^{-1}x) + \frac{d}{dx}(\sqrt{1-x^2})$

$$= (x)\left(\frac{1}{\sqrt{1-x^2}}\right) + (\sin^{-1}x)(1) + \frac{1}{2\sqrt{1-x^2}}(-2x)$$
$$= \sin^{-1}x$$

[redacted]

**9.** $x(t) = \sin^{-1}\left(\frac{t}{4}\right)$

$$y = \sin^{-1}u \qquad u = \frac{t}{4}$$
$$\sin y = u \qquad \frac{du}{dt} = \frac{1}{4}$$
$$\frac{d}{du}(\sin y) = \frac{d}{du}u$$
$$\cos y\frac{dy}{du} = 1$$
$$\frac{dy}{du} = \frac{1}{\cos y}$$
$$\frac{d}{du}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$v(t) = \frac{1}{4\sqrt{1-t^2/16}}$$
$$v(3) = \frac{1}{4\sqrt{1-9/16}} = \frac{\sqrt{7}}{7}$$

[redacted]

**11.** $x(t)=\tan^{-1} t$

$$y=\tan^{-1} t$$

$$\frac{d}{dt}\tan y=\frac{d}{dt}t$$

$$\sec^2 y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+(\tan y)^2}$$

$$=\frac{1}{1+t^2}$$

$$x(2)=\frac{1}{1+2^2}$$

$$=\frac{1}{5}$$

[redacted]

**13.** $\frac{dy}{ds}=\frac{d}{ds}\sec^{-1}(2s+1)$

$$=\frac{1}{|2s+1|\sqrt{(2s+1)^2-1}}\frac{d}{ds}(2s+1)$$

$$=\frac{1}{|2s+1|\sqrt{4s^2+4s}}(2)=\frac{1}{|2s+1|\sqrt{s^2+s}}$$

[redacted]

**15.** $\frac{dy}{dx}=\frac{d}{dx}\csc^{-1}(x^2+1)$

$$=-\frac{1}{|x^2+1|\sqrt{(x^2+1^2-1}}\frac{d}{dx}(x^2+1)$$

$$=-\frac{2x}{(x^2+1)\sqrt{x^4+2x^2}}=-\frac{2}{(x^2+1)\sqrt{x^2+2}}$$

Note that the condition $x>0$ is required in the last step.

[redacted]

**17.** $\frac{dy}{dt}=\frac{d}{dt}\sec^{-1}\left(\frac{1}{t}\right)=\frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\frac{d}{dt}\left(\frac{1}{t}\right)$

$$=\frac{1}{\left|\frac{1}{t}\right|\sqrt{\left(\frac{1}{t}\right)^2-1}}\left(-\frac{1}{t^2}\right)=-\frac{1}{\sqrt{1-t^2}}$$

Note that the condition $t>0$ is required in the last step.

[redacted]

**19.** $\frac{dy}{dt}=-\frac{d}{dt}\cot^{-1}\sqrt{t-1}=-\frac{1}{1+(\sqrt{t-1})^2}\frac{d}{dt}\sqrt{t-1}$

$$=-\left(\frac{1}{1+t-1}\right)\left(\frac{1}{2\sqrt{t-1}}\right)=-\frac{1}{2t\sqrt{t-1}}$$

[redacted]

**21.** $\frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}\sqrt{x^2-1})+\frac{d}{dx}(\csc^{-1} x)$

$$=\frac{1}{1+(\sqrt{x^2-1})^2}\frac{d}{dx}(\sqrt{x^2-1})-\frac{1}{|x|\sqrt{x^2-1}}$$

$$=\frac{1}{x^2}\frac{1}{2\sqrt{x^2-1}}(2x)-\frac{1}{|x|\sqrt{x^2-1}}$$

$$=\frac{1}{x\sqrt{x^2-1}}-\frac{1}{|x|\sqrt{x^2-1}}$$

$$=0$$

Note that the condition $x>1$ is required in the last step.

[redacted]

**23.** $y = \sec^{-1} x$

$$\frac{dy}{dx}\sec y = \frac{d}{dx}x$$

$$\sec y \ \tan y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \ \tan y}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$y'(2) = \frac{1}{2\sqrt{4-1}} = \frac{1}{2\sqrt{3}} = 0.289$$

$$y(2) = \sec^{-1} 2 = 2.203$$

$$y = 0.289(x-2) + 2.203$$

$$y = 0.289x + 1.625$$

[redacted]

**25.** $y = \sin^{-1}\left(\frac{x}{4}\right)$

$y = \sin^{-1} u \qquad u = \frac{x}{4}$

$\sin y = u \qquad \frac{du}{dx} = \frac{1}{4}$

$$\frac{d}{du}(\sin y) = \frac{d}{d}u$$

$$\cos y\frac{dy}{du} = 1$$

$$\frac{dy}{du} = \frac{1}{\cos y}$$

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$y' = \frac{1}{4\sqrt{1-\frac{x^2}{16}}}$$

$$y'(3) = \frac{1}{4\sqrt{1-\frac{9}{16}}}$$

$$y'(3) = 0.378$$

$$y(3) = \sin^{-1}\left(\frac{3}{4}\right) = 0.848$$

$$y = 0.378(x-3) + 0.848$$

$$y = 0.378x - 0.286$$

[redacted]

**27.** **(a)** Since $\frac{dy}{dx} = \sec^2 x$, the slope at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\left(\frac{\pi}{4}\right) = 2$.

The tangent line is given by

$y = 2\left(x - \frac{\pi}{4}\right) + 1$, or $y = 2x = \frac{\pi}{2} + 1$.

**(b)** Since $\frac{dy}{dx} = \frac{1}{1+x^2}$, the slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{1}{1+1^2} = \frac{1}{2}$.

The tangent line is given by $x \neq 0$.

[redacted]

**29. (a)** Note that $f'(x) = -\sin x + 3$, which is always between 2 and 4. Thus $f$ is differentiable at every point on the interval $(-\infty, \infty)$ and $f'(x)$ is never zero on this interval, so $f$ has a differentiable inverse by Theorem 3.

**(b)** $f(0) = \cos 0 + 3(0) = 1$;
$f'(0) = -\sin 0 + 3 = 3$

**(c)** Since the graph of $y = f(x)$ includes the point (0, 1) and the slope of the graph is 3 at this point, the graph of $y = f^{-1}(x)$ will include (1, 0) and the slope will be $\frac{1}{3}$,

Thus, $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \frac{1}{3}$.

[redacted]

**31. (a)** $v(t) = \frac{dx}{dt} = \frac{1}{1+t^2}$ which is always positive.

**(b)** $a(t) = \frac{dv}{dt} = -\frac{2t}{(1+t^2)^2}$ which is always negative.

**(c)** $\frac{\pi}{2}$

**32.** $\frac{d}{dx}\cos^{-1}(x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right)$

$$= 0 - \frac{d}{dx}\sin^{-1}(x)$$
$$= -\frac{1}{\sqrt{1-x^2}}$$

**33.** $\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1}(x)\right)$

$$= 0 - \frac{d}{dx}\tan^{-1}(x)$$
$$= -\frac{1}{1+x^2}$$

[redacted]

**35.** True. By definition of the Function.

[redacted]

**37.** E. $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)$

$$y = \sin^{-1} x \qquad u = \frac{x}{2}$$
$$\sin y = x \qquad \frac{du}{dx} = \frac{1}{2}$$
$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$
$$\cos y\frac{dy}{dx} = 1$$
$$\frac{d}{dx} = \frac{1}{\cos y}$$
$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-\frac{x}{2}}}\frac{1}{2}$$
$$= \frac{1}{\sqrt{4-x^2}}$$

[redacted]

**39.** A. $\frac{d}{dx}\sec^{-1}(x^2)$

$$y = \sec^{-1} u \qquad u = x^2$$

$$\frac{d}{du}(\sec y) = \frac{d}{du}u \qquad du = 2x$$

$$\sec y \tan y \frac{dy}{du} = 1$$

$$\frac{dy}{du} = \frac{1}{\sec y \tan y}$$

$$\frac{dy}{du} = \frac{1}{4\sqrt{u^2-1}}\frac{du}{dx}$$

$$= \frac{2x}{x^2\sqrt{x^4-1}}$$

$$= \frac{2}{x\sqrt{x^4-1}}$$

[redacted]

**41.** **(a)** $y = \frac{\pi}{2}$

**(b)** $y = -\frac{\pi}{2}$

**(c)** None, since $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \neq 0$.

[redacted]

**43.** **(a)** $y = \frac{\pi}{2}$

**(b)** $y = \frac{\pi}{2}$

**(c)** None, since $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \neq 0$.

[redacted]

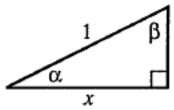
**45.** **(a)** None, since $\sin^{-1} x$ is undefined for $x > 1$.

**(b)** None, since $\sin^{-1} x$ is undefined for $x < -1$.

**(c)** None, since $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \neq 0$.

[redacted]

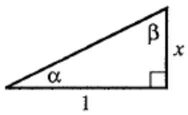
**47.** **(a)**




$\alpha = \cos^{-1} x, \beta = \sin^{-1} x$

So $\cos^{-1} x + \sin^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

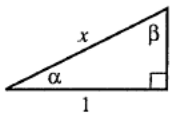
**(b)**




$\alpha = \tan^{-1} x, \beta = \cot^{-1} x$

So $\tan^{-1} x + \cot^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

**(c)**

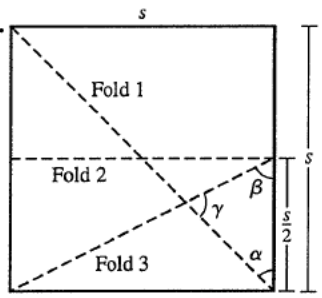



$\alpha = \sec^{-1} x, \beta = \csc^{-1} x$

So $\sec^{-1} x + \csc^{-1} x = \alpha + \beta = \frac{\pi}{2}$.

[redacted]

**49.**




If $s$ is the length of a side of the square, then

$\tan\alpha = \frac{s}{s} = 1$, so $\alpha = \tan^{-1} 1$ and

$\tan\beta = \frac{s}{\frac{s}{2}} = 2$, so $\beta = \tan^{-1} 2$.

From Exercise 34, we have

$\gamma = \pi - \alpha - \beta = \pi - \tan^{-1} 1 - \tan^{-1} 2 = \tan^{-1} 3.$

## **Section 3.9** Derivatives of Exponential and Logarithmic Functions (pp. 172–180)

### **Exploration 1 Leaving Milk on the Counter**

1. The temperature of the refrigerator is 42°F, the temperature of the milk at time $t = 0$.
2. The temperature of the room is 72°F, the limit to which $y$ tends as $t$ increases.
3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2(0.98)^t$ is negative, so $y'$ (the rate at which the milk is warming) is maximized at the lowest value of $t$.
4. We set $y = 55$ and solve;

$$72 - 30(0.98)^t = 55$$
$$(0.98)^t = \frac{17}{30}$$
$$t\ln(0.98) = \ln\left(\frac{17}{30}\right)$$
$$t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)} = 28.114$$

The milk reaches a temperature of 55° F after about 28 minutes.

5. $\frac{dy}{dx} = -30\ln(0.98)\cdot(0.98)^t$. At $t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)}$,

$\frac{dy}{dx} \approx 0.343$ degrees/minute.

### **Quick Review 3.9**

1. $\log_5 8 = \frac{\ln 8}{\ln 5}$
2. $7^x = e^{\ln 7^x} = e^{x\ln 7}$
3. $\ln(e^{\tan x}) = \tan x$
4. $\ln(x^2 - 4) - \ln(x+2) = \ln\frac{x^2-4}{x+2}$
$= \ln\frac{(x+2)(x-2)}{x+2} = \ln(x-2)$
5. $\log_2(8^{x-5}) = \log_2(2^3)^{x-5} = \log_2 2^{3x-15} = 3x - 15$
6. $\frac{\log_4 x^{15}}{\log_4 x^{12}} = \frac{15\log_4 x}{12\log_4 x} = \frac{15}{12} = \frac{5}{4}, x > 0$
7. $3\ln x - \ln 3x + \ln(12x^2) = \ln x^3 - \ln 3x + \ln(12x^2)$
$= \ln\frac{(x^3)(12x^2)}{3x} = \ln(4x^4)$
8. 
$$3^x = 19$$
$$\ln 3^x = \ln 19$$
$$x\ln 3 = \ln 19$$
$$x = \frac{\ln 19}{\ln 3} \approx 2.68$$
9. 
$$5^t \ln 5 = 18$$
$$5^t = \frac{18}{\ln 5}$$
$$\ln 5^t = \ln\frac{18}{\ln 5}$$
$$t\ln 5 = \ln 18 - \ln(\ln 5)$$
$$t = \frac{\ln 18 - \ln(\ln 5)}{\ln 5} \approx 1.50$$
10. 
$$3^{x+1} = 2x$$
$$\ln 3^{x+1} = \ln 2^x$$
$$(x+1)\ln 3 = x\ln 2$$
$$x(\ln 3 - \ln 2) = -\ln 3$$
$$x = \frac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$$

### **Section 3.9** Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}(2e^x) = 2e^x$

[redacted]

3. $\frac{dy}{dx} = \frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = -e^{-x}$

[redacted]

5. $\frac{dy}{dx} = \frac{d}{dx}e^{2x/3} = e^{2x/3}\frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}e^{2x/3}$

[redacted]

7. $\frac{dy}{dx} = \frac{d}{dx}(xe^2) - \frac{d}{dx}(e^x) = e^2 - e^x$

[redacted]

**9.** $\frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x}) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

[redacted]

**11.** $\frac{dy}{dx} = \frac{d}{dx} 8^x = 8^x \ln 8$

[redacted]

**13.** $\frac{dy}{dx} = \frac{d}{dx} 3^{\csc x} = 3^{\csc x}(\ln 3)\frac{d}{dx}(\csc x)$

$= 3^{\csc x}(\ln 3)(-\csc x \cot x)$

$= -3^{\csc x}(\ln 3)(\csc x \cot x)$

[redacted]

**15.** $\frac{dy}{dx} = \frac{d}{dx} \ln(x^2) = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{1}{x^2}(2x) = \frac{2}{x}$

[redacted]

**17.** $\frac{dy}{dx} = \frac{d}{dx} \ln(x^{-1}) = \frac{d}{dx}(-\ln x) = -\frac{1}{x},\ x > 0$

[redacted]

$= -\frac{1}{x}, x > 0$

**19.** $\frac{d}{dx} \ln(\ln x) = \frac{1}{\ln x}\frac{d}{dx} \ln x = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$

[redacted]

$= 1 + \ln x - 1 = \ln x$

**21.** $\frac{dy}{dx} = \frac{d}{dx}(\log_4 x^2) = \frac{d}{dx}\frac{\ln x^2}{\ln 4} = \frac{d}{dx}\left[\left(\frac{2}{\ln 4}\right)(\ln x)\right]$

$= \frac{2}{\ln 4} \cdot \frac{1}{x} = \frac{2}{x \ln 4} = \frac{1}{x \ln 2}$

[redacted]

**23.** $\frac{dy}{dx} = \frac{d}{dx} \log_2\left(\frac{1}{x}\right) = \frac{d}{dx}(-\log_2 x) = -\frac{1}{x \ln 2}, x > 0$

[redacted]

**25.** $\frac{dy}{dx} = \frac{d}{dx}(\ln 2 \cdot \log_2 x) = (\ln 2)\frac{d}{dx}(\log_2 x)$

$= (\ln 2)\left(\frac{1}{x \ln 2}\right) = \frac{1}{x}, x > 0$

[redacted]

**27.** $\frac{dy}{dx} = \frac{d}{dx}(\log_{10} e^x) = \frac{d}{dx}(x \log_{10} e) = \log_{10} e = \frac{\ln e}{\ln 10}$

$= \frac{1}{\ln 10}$

[redacted]

**29.** $m = 5$

$y = 3^x + 1$

$y' = 3^x \ln 3 = 5$

$x = 1.379$

$y = 3^{1.379} + 1 = 5.551$

$(1.379,\ 5.551)$

[redacted]

**31.** $y = \ln 2x$

$e^y = 2x$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(2x)$

$e^y \frac{dy}{dx} = 2$

$\frac{dy}{dx} = 2e^{-1}$

[redacted]

33. $\frac{dy}{dx} = \frac{d}{dx}(x^{\pi}) = \pi x^{\pi-1}$

[redacted]

35. $\frac{dy}{dx} = \frac{d}{dx} x^{-\sqrt{2}} = -\sqrt{2x}^{-\sqrt{2}-1}$

[redacted]

37. $\frac{d}{dx} \ln(x+2) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx} \ln(u) \qquad u = x+2$

$f'(x) = \frac{1}{x+2} \qquad \frac{du}{dx} = 1$

$x+2 > 0$

$x > -2$

[redacted]

39. $\frac{d}{dx} \ln(2-\cos x) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx} \ln(u) \qquad u = 2-\cos x$

$\frac{du}{dx} = \sin x$

$f'(x) = \frac{\sin x}{2-\cos x}$

Domain of $x$ is all real numbers.

[redacted]

41. $\frac{d}{dx} \log_2(3x+1) = \frac{1}{4 \ln a}\frac{du}{dx}$

$\frac{d}{dx} \log_2(u) \qquad u = 3x+1$

$a = 2 \qquad \frac{du}{dx} = 3$

$f'(x) = \frac{3}{(3x+1)\ln 2}$

$3x+1 > 0$

$x > -1/3$

[redacted]

43.
$$y = (\sin x)^x$$
$$\ln y = \ln(\sin x)^x$$
$$\ln y = x \ln(\sin x)$$
$$\frac{d}{dx} \ln y = \frac{d}{dx}\left[x \ln(\sin x)\right]$$
$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1)$$
$$\frac{dy}{dx} = y\left[x \cot x + \ln(\sin x)\right]$$
$$\frac{dy}{dx} = (\sin x)^x [x \cot x + \ln(\sin x)]$$

[redacted]

**45.** $y = \sqrt[5]{\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\dfrac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$

$$\ln y = \ln\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$$

$$\ln y = \frac{1}{5}\ln\frac{(x-3)^4(x^2+1)}{(2x+5)^3}$$

$$\ln y = \frac{1}{5}[4\ln(x-3) + \ln(x^2+1) - 3\ln(2x+5)]$$

$$\frac{d}{dx}(\ln y) = \frac{4}{5}\frac{d}{dx}\ln(x-3) + \frac{1}{5}\frac{d}{dx}\ln(x^2+1) - \frac{3}{5}\frac{d}{dx}\ln(2x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{5}\frac{1}{x-3} + \frac{1}{5}\frac{1}{x^2+1}(2x) - \frac{3}{5}\frac{1}{2x+5}(2)$$

$$\frac{dy}{dx} = y\left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right)$$

$$\frac{dy}{dx} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5} \cdot \left(\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x+5)}\right)$$

[redacted]

**47.** $y = x^{\ln x}$

$$\ln y = \ln x^{\ln x}$$

$$\ln y = \ln x \ln x$$

$$\frac{dy}{dx}(\ln y) = \frac{d}{dx}(\ln x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x}$$

$$\frac{dy}{dx} = \frac{2y\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}$$

[redacted]

**49.** The line passes through $(a, e^a)$ for some value of $a$ and has slope $m = e^a$. Since the line also passes through the origin, the slope is also given by $e$, $e\ln x < x$ or $\ln x^e < x$. and we have $e^a = \dfrac{e^a}{a}$, so $a = 1$. Hence, the slope is $e$ and the equation is $y = ex$.

[redacted]

**51. (a)** $P(0) = \dfrac{300}{1+2^{4-0}} \approx 18$

**(b)** for all positive $x \neq e$.

$$= 300\frac{d}{dt}(1+2^{4-t})^{-1}$$

$$= 300\left(\frac{16\ln(2)2^t}{(2^t+16)^2}\right)$$

$$P'(4) = \frac{4800\ln(2)2^4}{(2^4+16)^2} = 52$$

**(c)** After 4 days, the rumor will spread to 52 $\dfrac{\text{students}}{\text{day}}$.

[redacted]

**53.** $\frac{dA}{dt} = 20\frac{d}{dt}\left(\frac{1}{2}\right)^{t/140}$

$= 20\frac{d}{dt}2^{-t/140}$

$= 20(2^{-t/140})(\ln 2)\frac{d}{dt}\left(-\frac{t}{140}\right)$

$= 20(2^{-t/140})(\ln 2)\left(-\frac{t}{140}\right)$

$= -\frac{(2^{-t/140})(\ln 2)}{7}$

At $t = 2$ days, we have $\frac{dA}{dt} = -\frac{(2^{-1/70})(\ln 2)}{7} \approx -0.098$ grams/day.

This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

[redacted]

**55.** **(a)** Since $f'(x) = 2^x \ln 2, f'(0) = 2^0 \ln 2 = \ln 2$.

**(b)** $f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{2^h - 2^0}{h}$

$= \lim_{h\to 0}\frac{2^h - 1}{h}$

**(c)** Since quantities in parts (a) and (b) are equal,

$\lim_{h\to 0}\frac{2^h - 1}{h} = \ln 2.$

**(d)** By following the same procedure as above using $g(x) = 7^x$, we may see that $\lim_{h\to 0}\frac{7^h - 1}{h} = \ln 7$.

[redacted]

**57.** False. It is $\ln(2)2^x$.

[redacted]

**59.** B. $P(0) = \frac{150}{1 + e^{4-t}} = 3$

[redacted]

**61.** A. $y = \log_{10}(2x - 3)$

$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$

$a = 10$

$u = 2x - 3$

$\frac{du}{dx} = 2$

$y' = \frac{2}{(2x - 3)\ln 10}$

[redacted]

**63.** **(a)** The graph $y_4$ is a horizontal line at $y = a$.

**(b)** The graph of $y_3$ is always a horizontal line.

| $a$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $y_3$ | 0.693147 | 1.098613 | 1.386295 | 1.609439 |
| $\ln a$ | 0.693147 | 1.098612 | 1.386294 | 1.609438 |

We conclude that the graph of $y_3$ is a horizontal line at $y = \ln a$.

**(c)** $\frac{d}{dx}a^x = a^x$ if and only if $y_3 = \frac{y_2}{y_1} = 1$.

So if $y_3 = \ln a$, then $\frac{d}{dx}a^x$ with equal $a^x$ if and only if $\ln a = 1$, or $a = e$.

**(d)** $y_2 = \frac{d}{dx}a^x = a^x \ln a$. This will equal $y_1 = a^x$ if and only if $\ln a = 1$, or $a = e$.

[redacted]

**65.** **(a)** Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

**(b)** The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

**(c)** Multiplying by $e$, $e \ln x < x$ or $\ln x^e < x$.

**(d)** Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

**(e)** Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, $e^\pi$ is bigger.

### Quick Quiz Sections 3.7–3.9

**1.** E. $y = \frac{9}{2x} - \frac{x^2}{2}$

$dy = -\frac{9}{2x^2} - x$

$dy = -\frac{9}{2(1)^2} - 1 = -\frac{11}{2}$

**2.** A. $dy = \frac{d}{dx}(\cos^3(3x-2))$

$dy = -9\cos^2(3x-2)\sin(3x-2)$

**3.** C. $dy = \frac{d}{dx}(\sin^{-1}(2x))$

$dy = \frac{2}{\sqrt{1-4x^2}}$

**4.** **(a)** Differentiate implicitly:

$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$

$1 \cdot y^2 + x \cdot 2y\frac{dy}{dx} - \left(3x^2y + x^3\frac{dy}{dx}\right) = 0$

$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$

$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

**(b)** If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

at $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)^2(-2) - (-2)^2}{2(1)(-2) - (1)^3} = 2.$

The tangent line is $y + 2 = 2(x - 1)$.

At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)^2(3) - 3^2}{2(1)(3) - 1^3} = 0.$

The tangent line is $y = 3$.

**(c)** The tangent line is vertical where $2xy - x^3 = 0$, which implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve where $x = 0$. If $y = \frac{x^2}{2}$, then $x\left(\frac{x^2}{2}\right) - x^3\left(\frac{x^2}{2}\right) = 6.$

Then the only solution to this equation is $x = \sqrt[5]{-24}$.

## Chapter 3 Review Exercises
## (pp. 181–184)

**1.** $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

**2.** $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

**3.** $\frac{dy}{dx} = \frac{d}{dx}(2\sin x\cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2\sin^2 x + 2\cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2\sin x\cos x) = \frac{d}{dx}\sin 2x = (\cos 2x)(2)$

$= 2\cos 2x$

**4.** $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+1}{2x-1} = \frac{(2x-1)(2) - (2x-1)(2)}{(2x-1)^2} = -\frac{4}{(2x-1)^2}$

**5.** $\frac{ds}{dt} = \frac{d}{dt}\cos(1-2t) = -\sin(1-2t)(-2) = 2\sin(1-2t)$

**6.** $\frac{ds}{dt} = \frac{d}{dt}\cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2}\csc^2\left(\frac{2}{t}\right)$

**7.** $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

**8.** $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x+1}) = (x)\left(\frac{1}{2\sqrt{2x+1}}\right)(2) + (\sqrt{2x+1})(1)$

$= \frac{x + (2x+1)}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$

**9.** $\frac{dr}{d\theta} = \frac{d}{d\theta}\sec(1+3\theta) = \sec(1+3\theta)\tan(1+3\theta)(3)$

$= 3\sec(1+3\theta)\tan(1+3\theta)$

**10.** $\frac{dr}{d\theta} = \frac{d}{d\theta}\tan^2(3-\theta^2)$

$= 2\tan(3-\theta^2)\frac{d}{d\theta}\tan(3-\theta^2)$

$= 2\tan(3-\theta^2)\sec^2(3-\theta^2)(-2\theta)$

$= -4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)$

**11.** $\frac{dy}{dx} = \frac{d}{dx}(x^2\csc 5x)$

$= (x^2)(-\csc 5x\cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2\csc 5x\cot 5x + 2x\csc 5x$

**12.** $\frac{dy}{dx}=\frac{d}{dx}\ln\sqrt{x}=\frac{1}{\sqrt{x}}\frac{d}{dx}\sqrt{x}=\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2x},\ x>0$

**13.** $\frac{dy}{dx}=\frac{d}{dx}\ln(1+e^x)=\frac{1}{1+e^x}\frac{d}{dx}(1+e^x)=\frac{e^x}{1+e^x}$

**14.** $\frac{dy}{dx}=\frac{d}{dx}(xe^{-x})=(x)(e^{-x})(-1)+(e^{-x})(1)=-xe^{-x}+e^{-x}$

**15.** $\frac{dy}{dx}=\frac{d}{dx}(e^{1+\ln x})=\frac{d}{dx}(e^1e^{\ln x})=\frac{d}{dx}(ex)=e$

**16.** $\frac{dy}{dx}=\frac{d}{dx}\ln(\sin x)=\frac{1}{\sin x}\frac{d}{dx}(\sin x)=\frac{\cos x}{\sin x}=\cot x$, for values of $x$ in the intervals $(k\pi,\ (k+1)\pi)$, where $k$ is even.

**17.** $\frac{dr}{dx}=\frac{d}{dx}\ln(\cos^{-1}x)=\frac{1}{\cos^{-1}x}\frac{d}{dx}\cos^{-1}x$

$$=\frac{1}{\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right)=-\frac{1}{\cos^{-1}x\sqrt{1-x^2}}$$

**18.** $\frac{dr}{d\theta}=\frac{d}{d\theta}\log_2(\theta^2)=\frac{1}{\theta^2\ln 2}\frac{d}{d\theta}(\theta^2)=\frac{2\theta}{\theta^2\ln 2}=\frac{2}{\theta\ln 2}$

**19.** $\frac{ds}{dt}=\frac{d}{dt}\log_5(t-7)=\frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7)=\frac{1}{(t-7)\ln 5}$, $t>7$

**20.** $\frac{ds}{dt}=\frac{d}{dt}(8^{-t})=8^{-t}(\ln 8)\frac{d}{dt}(-t)=-8^{-t}\ln 8$

**21.** Use logarithmic differentiation.

$$y=x^{\ln x}$$
$$\ln y=\ln(x^{\ln x})$$
$$\ln y=(\ln x)(\ln x)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}(\ln x)^2$$
$$\frac{1}{y}\frac{dy}{dx}=2\ln x\frac{d}{dx}\ln x$$
$$\frac{dy}{dx}=\frac{2y\ln x}{x}$$
$$\frac{dy}{dx}=\frac{2x^{\ln x}\ln x}{x}$$

**22.** $\frac{dy}{dx}=\frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}}$

$$=\frac{\sqrt{x^2+1}\frac{d}{dx}\left[(2x)2^x\right]-(2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1}$$

$$=\frac{\sqrt{x^2+1}\left[(2x)(2^x)(\ln 2)+(2^x)(2)\right]-(2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$=\frac{(x^2+1)(2^x)(2x\ln 2+2)-2x^2(2^x)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)[(x^2+1)(x\ln 2+1)-x^2]}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)(x^3\ln 2+x^2+x\ln 2+1-x^2)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)(x^3\ln 2+x\ln 2+1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y=\frac{(2x)2^x}{\sqrt{x^2+1}}$$
$$\ln y=(2x)+\ln(2^x)-\ln\sqrt{x^2+1}$$
$$\ln y=\ln 2+\ln x+x\ln 2-\frac{1}{2}\ln(x^2+1)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}[\ln 2+\ln x+\ln 2-\frac{1}{2}\ln(x^2+1)]$$
$$\frac{1}{y}\frac{dy}{dx}=0+\frac{1}{x}+\ln 2-\frac{1}{2}\frac{1}{x^2+1}(2x)$$
$$\frac{dy}{dx}=y\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$
$$\frac{dy}{dx}=\frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$

**23.** $\frac{dy}{dx}=\frac{d}{dx}e^{\tan^{-1}x}=e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x=\frac{e^{\tan^{-1}x}}{1+x^2}$

**24.** $\frac{dy}{du}=\frac{d}{dx}\sin^{-1}\sqrt{1-u^2}$

$$=\frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\frac{d}{du}\sqrt{1-u^2}$$

$$=\frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u)=\frac{u}{|u|\sqrt{1-u^2}}$$

**25.** $\frac{dy}{dt}=\frac{d}{dt}\left(t\sec^{-1}t-\frac{1}{2}\ln t\right)$

$$=(t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right)+(\sec^{-1}t)(1)-\frac{1}{2t}$$

$$=\frac{t}{|t|\sqrt{t^2-1}}+\sec^{-1}t-\frac{1}{2t}$$

**26.** $\frac{dy}{dt}=\frac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$$=(1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2)+(\cot^{-1}2t)(2t)$$

$$=-\frac{2+2t^2}{1+4t^2}+2t\cot^{-1}2t$$

**27.** $\frac{dy}{dz}=\frac{d}{dz}(z\cos^{-1}z-\sqrt{1-z^2})$

$$=(z)\left(-\frac{1}{\sqrt{1-z^2}}\right)+(\cos^{-1}z)(1)-\frac{1}{2\sqrt{1-z^2}}(-2z)$$
$$=-\frac{z}{\sqrt{1-z^2}}+\cos^{-1}z+\frac{z}{\sqrt{1-z^2}}$$
$$=\cos^{-1}z$$

**28.** $\frac{dy}{dx}=\frac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$$=(2\sqrt{x-1})\left(-\frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\right)\left(\frac{1}{2\sqrt{x}}\right)+(2\csc^{-1}\sqrt{x})\left(\frac{1}{2\sqrt{x-1}}\right)$$
$$=-\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}}+\frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$
$$=-\frac{1}{x}+\frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$$

**29.** $\frac{dy}{dx}=\frac{d}{dx}\csc^{-1}(\sec x)$

$$=\left(-\frac{1}{|\sec x|\sqrt{\sec^2x-1}}\right)\frac{d}{dx}(\sec x)$$
$$=-\frac{1}{|\sec x|\sqrt{\tan^2x-1}}\sec x\tan x$$
$$=-\frac{\sec x\tan x}{|\sec x\tan x|}$$
$$=-\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|}=-\frac{\sin x}{|\sin x|}$$
$$=\begin{cases}-1, & 0\le x<\pi, \quad x\ne\frac{\pi}{2}\\ 1, & \pi<x\le 2\pi, \quad x\ne\frac{3\pi}{2}\end{cases}$$

Alternate method:

On the domain $0\le x\le 2\pi,\ x\ne\frac{\pi}{2},\ x\ne\frac{3\pi}{2}$, we may rewrite the function as follows:

$$y=\csc^{-1}(\sec x)$$
$$=\frac{\pi}{2}-\sec^{-1}(\sec x)$$
$$=\frac{\pi}{2}-\cos^{-1}(\cos x)$$
$$=\begin{cases}\frac{\pi}{2}-x, & 0\le x\le\pi, \quad x\ne\frac{\pi}{2}\\ \frac{\pi}{2}-(\pi-x), & \pi<x\le 2\pi, \quad x\ne\frac{3\pi}{2}\end{cases}$$
$$=\begin{cases}\frac{\pi}{2}-x, & 0\le x\le\pi, \quad x\ne\frac{\pi}{2}\\ \frac{\pi}{2}-(\pi+x), & \pi<x\le 2\pi, \quad x\ne\frac{3\pi}{2}\end{cases}$$

Therefore, $\frac{dy}{dx}=\begin{cases}-1, & 0\le x<\pi, \quad x\ne\frac{\pi}{2}\\ 1, & \pi<x\le 2\pi, \quad x\ne\frac{3\pi}{2}\end{cases}$

Note that the derivative exists at 0 and $2\pi$ only because these are the endpoints of the given domain; the two-sided derivative of $y=\csc^{-1}(\sec x)$ does not exist at these points.

**30.** $\frac{dr}{d\theta}=\frac{d}{d\theta}\left(\frac{1+\sin\theta}{1-\cos\theta}\right)^2$

$$=2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{(1-\cos\theta)(\cos\theta)-(1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}\right)$$
$$=2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta-\cos^2\theta-\sin\theta-\sin^2\theta}{(1-\cos\theta)^2}\right)$$
$$=2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta-\sin\theta-1}{(1-\cos\theta)^2}\right)$$

**31.** Since $y=\ln x^2$ is defined for all $x\ne 0$ and $\frac{dy}{dx}=\frac{1}{x^2}\frac{d}{dx}(x^2)=\frac{2x}{x^2}=\frac{2}{x}$, the function is differentiable for all $x\ne 0$.

**32.** Since $y=\sin x-x\cos x$ is defined for all real $x$ and $\frac{dy}{dx}=\cos x-(x)(-\sin x)-(\cos x)(1)=x\sin x$, the function is differentiable for all real x.

**33.** Since $y=\sqrt{\frac{1-x}{1+x^2}}$ is defined for all $x<1$ and

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{1-x}{1+x^2}}}\cdot\frac{(1+x^2)(-1)-(1-x)(2x)}{(1+x^2)^2}$$
$$=\frac{x^2-2x-1}{2\sqrt{1-x}(1+x^2)^{3/2}},$$ which is defined only for $x<1$, the function is differentiable for all $x<1$.

**34.** Since $y=(2x-7)^{-1}(x+5)=\frac{x+5}{2x-7}$ is defined for all $x\ne\frac{7}{2}$ and $\frac{dy}{dx}=\frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2}=-\frac{17}{(2x-7)^2}$, the function is differentiable for all $x\ne\frac{7}{2}$.

**35.** Use implicit differentiation.

$$xy+2x+3y=1$$
$$\frac{dy}{dx}(xy)+\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(1)$$
$$x\frac{d}{dx}+(y)(1)+2+3\frac{dy}{dx}=0$$
$$(x+3)\frac{dy}{dx}=-(y+2)$$
$$\frac{dy}{dx}=-\frac{y+2}{x+3}$$

**36.** Use implicit differentiation.

$$5x^{4/5}+10y^{6/5}=15$$
$$\frac{d}{dx}(5x^{4/5})+\frac{d}{dx}(10y^{6/5})=\frac{d}{dx}(15)$$
$$4x^{-1/5}+12y^{1/5}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{4x^{-1/5}}{12y^{1/5}}=-\frac{1}{3(xy)^{1/5}}$$

**37.** Using implicit differentiation.

$$\sqrt{xy}=1$$
$$\frac{d}{dx}\sqrt{xy}=\frac{d}{dx}(1)$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx}+(y)(1)\right]=0$$
$$x\frac{dy}{dx}+y=0$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

Alternate method:

**38.** Use implicit differentiation.

$$y^2=\frac{x}{x+1}$$
$$\frac{d}{dx}y^2=\frac{d}{dx}\frac{x}{x+1}$$
$$2y\frac{dy}{dx}=\frac{(x+1)(1)-(x)(1)}{(x+1)^2}$$
$$\frac{dy}{dx}=\frac{1}{2y(x+1)^2}$$

**39.**
$$x^3+y^3=1$$
$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=\frac{d}{dx}(1)$$
$$3x^2+3y^2y'=0$$
$$y'=-\frac{x^2}{y^2}$$
$$y''=\frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)(y')}{y^4}$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4}$$
$$=-\frac{2xy^3+2x^4}{y^5}$$
$$=-\frac{2x(x^3+y^3)}{y^5}$$
$$=-\frac{2x}{y^5}$$

since $x^3+y^3=1$

**40.**
$$y^2=1-\frac{2}{x}$$
$$\frac{d}{dx}(y^2)=\frac{d}{dx}(1)-\frac{d}{dx}\left(\frac{2}{x}\right)$$
$$2yy'=\frac{2}{x^2}$$
$$y'=\frac{2}{x^{2(2y)}}=\frac{1}{x^2y}$$
$$y''=\frac{d}{dx}\left(\frac{1}{x^2y}\right)$$
$$=-\frac{1}{(x^2y)^2}\frac{d}{dx}(x^2y)$$
$$=-\frac{1}{(x^2y)^2}[(x^2)(y')+(y)(2x)]$$
$$=-\frac{1}{(x^2y)^2}\left[(x^2)\left(\frac{1}{x^2y}\right)+2xy\right]$$
$$=-\frac{1}{x^4y^2}\left(\frac{1}{y}+2xy\right)$$
$$=-\frac{1+2xy^2}{x^4y^3}$$

**41.**
$$y^3+y=2\cos x$$
$$\frac{d}{dx}(y^3)+\frac{d}{dx}(y)=\frac{d}{dx}(2\cos x)$$
$$3y^2y'+y'=-2\sin x$$
$$(3y^2+1)y'=-2\sin x$$
$$y'=-\frac{2\sin x}{3y^2+1}$$
$$y''=\frac{d}{dx}\left(-\frac{2\sin x}{3y^2+1}\right)$$
$$=-\frac{(3y^2+1)(2\cos x)-(2\sin x)(6yy')}{(3y^2+1)^2}$$
$$=-\frac{(3y^2+1)(2\cos x)-(12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2+1)^2}$$
$$=-2\frac{(3y^2+1)^2\cos x+12y\sin^2 x}{(3y^2+1)^3}$$

**42.** $x^{1/3}+y^{1/3}=4$

$$\frac{d}{dx}(x^{1/3})+\frac{d}{dx}(y^{1/3})=\frac{d}{dx}(4)$$

$$\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0$$

$$y'=-\frac{x^{-2/3}}{y^{-2/3}}=-\left(\frac{y}{x}\right)^{2/3}$$

$$y''=\frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$

$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy'-(y)(1)}{x^2}\right)$$

$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right]-y}{x^2}\right)$$

$$=-\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3}-x^{-2}y)$$

$$=\frac{2}{3}x^{-4/3}y^{1/3}+\frac{2}{3}x^{-5/3}y^{2/3}$$

**43.** $y'=2x^3-3x-1,$

$y''=6x^2-3,$

$y'''=12x,$

$y^{(4)}=12$, and the rest are all zero.

**44.** $y'=\frac{x^4}{24},$

$y''=\frac{x^3}{6},$

$6y'''=\frac{x^2}{2},$

$y^{(4)}=x,$

$y^{(5)}=1$, and the rest are all zero.

**45.** $\frac{dy}{dx}=\frac{d}{dx}\sqrt{x^2-2x}=\frac{1}{2\sqrt{x^2-2x}}(2x-2)=\frac{x-1}{\sqrt{x^2-2x}}$

At $x=3$, we have $y=\sqrt{3^2-2(3)}=\sqrt{3}$

and $\frac{dy}{dx}=\frac{3-1}{\sqrt{3^2-2(3)}}=\frac{2}{\sqrt{3}}.$

**(a)** Tangent: $y=\frac{2}{\sqrt{3}}(x-3)+\sqrt{3}$ or $y=\frac{2}{\sqrt{3}}x-\sqrt{3}$

**(b)** Normal: $y=-\frac{\sqrt{3}}{2}(x-3)+\sqrt{3}$ or $y=-\frac{\sqrt{3}}{2}x+\frac{5\sqrt{3}}{2}$

**46.** $\frac{dy}{dx}=\frac{d}{dx}(4+\cot x-2\csc x)$

$=-\csc^2 x+2\csc x\cot x$

At $x=\frac{\pi}{2}$, we have

$y=4+\cot\frac{\pi}{2}-2\csc\frac{\pi}{2}=4+0-2=2$ and

$\frac{dy}{dx}=-\csc^2\frac{\pi}{2}+2\csc\frac{\pi}{2}\cot\frac{\pi}{2}=-1+2(1)(0)=-1.$

**(a)** Tangent: $y=-1\left(x-\frac{\pi}{2}\right)+2$ or $y=-x+\frac{\pi}{2}+2$

**(b)** Normal: $y=-1\left(x-\frac{\pi}{2}\right)+2$ or $y=x-\frac{\pi}{2}+2$

**47.** Use implicit differentiation.

$$x^2+2y^2=9$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(2y^2)=\frac{d}{dx}(9)$$

$$2x+4y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{4y}=-\frac{x}{2y}$$

Slope at (1, 2): $-\frac{1}{2(2)}=-\frac{1}{4}$

**(a)** Tangent: $y=-\frac{1}{4}(x-1)+2$ or $y=-\frac{1}{4}x+\frac{9}{4}$

**(b)** Normal: $y=4(x-1)+2$ or $y=4x-2$

**48.** Use implicit differentiation.

$$x+\sqrt{xy}=6$$

$$\frac{d}{dx}(x)+\frac{d}{dx}(\sqrt{xy})=\frac{d}{dx}(6)$$

$$1+\frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right)+(y)(1)\right]=0$$

$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx}=-1-\frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx}=\frac{2\sqrt{xy}}{x}\left(-1-\frac{y}{2\sqrt{xy}}\right)$$

$$=-2\sqrt{\frac{y}{x}}-\frac{y}{x}$$

Slope at (4, 1): $-2\sqrt{\frac{1}{4}}-\frac{1}{4}=-\frac{2}{2}-\frac{1}{4}=-\frac{5}{4}$

**(a)** Tangent: $y=-\frac{5}{4}(x-4)+1$ or $y=-\frac{5}{4}x+6$

**(b)** Normal: $y=-\frac{4}{5}(x-4)+1$ or $y=-\frac{4}{5}x-\frac{11}{5}$

**49.** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin t}{2\cos t} = -\tan t$

At $t = \frac{3\pi}{4}$, we have $x = 2\sin\frac{3\pi}{4} = \sqrt{2}$,

$y = 2\cos\frac{3\pi}{4} = -\sqrt{2}$, and $\frac{dy}{dx} = -\tan\frac{3\pi}{4} = 1$.

The equation of the tangent line is

$y = 1(x - \sqrt{2}) + (-\sqrt{2})$, or $y = x - 2\sqrt{2}$.

**50.** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos t}{-3\sin t} = -\frac{4}{3}\cot t$

At $t = \frac{3\pi}{4}$, we have $x = 3\cos\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$,

$y = 4\sin\frac{3\pi}{4} = 2\sqrt{2}$, and $\frac{dy}{dx} = -\frac{4}{3}\cot\frac{3\pi}{4} = \frac{4}{3}$.

The equation of the tangent line is

$y = \frac{4}{3}\left(x + \frac{3\sqrt{2}}{2}\right) + 2\sqrt{2}$, or $y = \frac{4}{3}x + 4\sqrt{2}$.

**51.** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5\sec^2 t}{3\sec t\tan t} = \frac{5\sec t}{3\tan t} = \frac{5}{3\sin t}$

At $t = \frac{\pi}{6}$, we have $x = 3\sec\frac{\pi}{6} = 2\sqrt{3}$,

$y = 5\tan\frac{\pi}{6} = \frac{5\sqrt{3}}{3}$, and $\frac{dy}{dx} = \frac{5}{3\sin\left(\frac{\pi}{6}\right)} = \frac{10}{3}$.

The equation of the tangent line is

$y = \frac{10}{3}(x - 2\sqrt{3}) + \frac{5\sqrt{3}}{3}$, or $y = \frac{10}{3}x - 5\sqrt{3}$.

**52.** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\cos t}{-\sin t}$

At $t = -\frac{\pi}{4}$, we have $x = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$y = -\frac{\pi}{4} + \sin\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4} - \frac{\sqrt{2}}{2}$, and

$\frac{dy}{dx} = \frac{1+\cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)} = \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2}+1$.

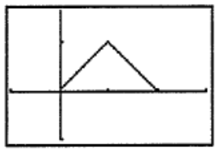
The equation of the tangent line is

$y = (\sqrt{2}+1)\left(x - \frac{\sqrt{2}}{2}\right) - \frac{\pi}{4} - \frac{\sqrt{2}}{2}$, or

$y = (1+\sqrt{2})x - \sqrt{2} - 1 - \frac{\pi}{4}$.

This is approximately $y = 2.414x - 3.200$.

**53. (a)**



[-1, 3] by [-1, 5/3]

**(b)** Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

**(c)** No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is $-1$.

**54. (a)** The function is continuous for all values of $m$, because the right-hand limit as $x \to 0$ is equal to $f(0) = 0$ for any value of $m$.

**(b)** The left-hand derivative at $x = 0$ is $2\cos(2 \bullet 0) = 2$, and the right-hand derivative at $x = 0$ is $m$, so in order for the function to be differentiable at $x = 0$, $m$ must be 2.

**55. (a)** For all $x \neq 0$ **(b)** At $x = 0$

**(c)** Nowhere

**56. (a)** For all $x$ **(b)** Nowhere

**(c)** Nowhere

**57.** Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x-3) = -3$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-3) = -3$. Since these values agree with $f(0)$, the function is continuous at $x = 0$. On the other hand, $f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is undefined at $x = 0$.

**(a)** $[-1, 0) \cup (0, 4]$ **(b)** At $x = 0$

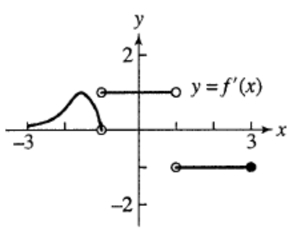
**(c)** Nowhere in its domain

**58.** Note that the function is undefined at $x = 0$.

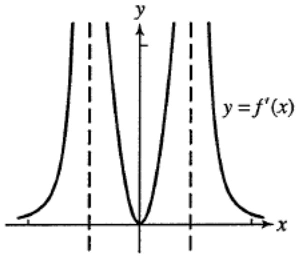
**(a)** $[-2, 0) \cup (0, 2]$ **(b)** Nowhere

**(c)** Nowhere in its domain

**59.**



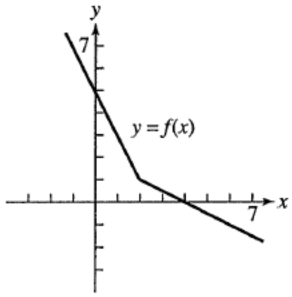
**60.**



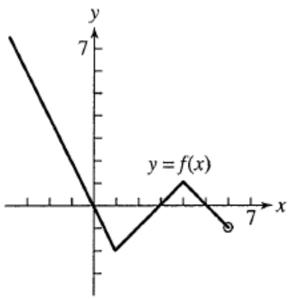

**61.** **(a)** iii **(b)** i

**(c)** ii

**62.** The graph passes through (0, 5) and has slope −2 for $x < 2$ and slope −0.5 for $x > 2$.




**63.** The graph passes through (−1, 2) and has slope −2 for $x < 1$, slope 1for $1 < x < 4$, and slope −1 for $4 < x < 6$.




**64.** **i.** If $f(x) = \frac{9}{28}x^{7/3} + 9$, then $f'(x) = \frac{3}{4}x^{4/3}$ and $f''(x) = x^{1/3}$, which matches the given equation.

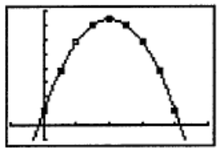
**ii.** If $f'(x) = \frac{9}{28}x^{7/3} - 2$, then $f''(x) = \frac{3}{4}x^{4/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

**iii.** If $f'(x) = \frac{3}{4}x^{4/3} + 6$, then $f''(x) = x^{1/3}$, which matches the given equation.

**iv.** If $f(x) = \frac{3}{4}x^{4/3} - 4$, then $f'(x) = x^{1/3}$ and $f''(x) = \frac{1}{3}x^{-2/3}$, which contradicts the given equation $f''(x) = x^{1/3}$.

Answer is D: i and iii only could be true. Note, however that i and iii could not simultaneously be true.

**65.** **(a)**

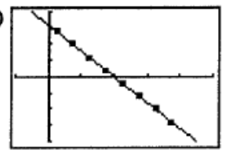


[-1, 5] by [-10,80]

**(b)**

| $t$ interval | avg. vel. |
|---|---|
| [0, 0.5] | $\frac{38-10}{0.5-0} = 56$ |
| [0.5, 1] | $\frac{58-38}{1-0.5} = 40$ |
| [1, 1.5] | $\frac{70-58}{1.5-1} = 24$ |
| [1.5, 2] | $\frac{74-70}{2-1.5} = 8$ |
| [2, 2.5] | $\frac{70-74}{2.5-2} = -8$ |
| [2.5, 3] | $\frac{58-70}{3-2.5} = -24$ |
| [3, 3.5] | $\frac{38-58}{3.5-3} = -40$ |
| [3.5, 4] | $\frac{10-38}{4-3.5} = -56$ |

**(c)**



[-1, 5] by [-80, 80]

**(d)** Average velocity is a good approximation to velocity.

**66.** **(a)** $\frac{d}{dx}\left[\sqrt{x}f(x)\right] = \sqrt{x}f'(x) + \frac{1}{2\sqrt{x}}f(x)$

At $x = 1$, the derivative is

$$\sqrt{1}f'(1) + \frac{1}{2\sqrt{1}}f(1) = 1\left(\frac{1}{5}\right) + \left(\frac{1}{2}\right)(-3) = -\frac{13}{10}.$$

**(b)** $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

At $x = 0$, the derivative is $\frac{f'(0)}{2\sqrt{f(0)}} = -\frac{2}{2\sqrt{9}} = -\frac{1}{3}$.

**(c)** $\frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x = 1$, the derivative is $\frac{f'(\sqrt{1})}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}$.

**(d)** $\frac{d}{dx}f(1 - 5\tan x) = f'(1 - 5\tan x)(-5\sec^2 x)$

At $x = 0$, the derivative is

$$f'(1 - 5\tan 0)(-5\sec^2 0) = f'(1)(-5) = \left(\frac{1}{5}\right)(-5) = -1.$$

**66. Continued**

**(e)** $\frac{d}{dx}\frac{f(x)}{2+\cos x} = \frac{(2+\cos x)(f'(x)) - (f(x))(-\sin x)}{(2+\cos x)^2}$

At $x = 0$, the derivative is

$$\frac{(2+\cos 0)(f'(0)) - (f(0))(-\sin 0)}{(2+\cos 0)^2} = \frac{3f'(0)}{3^2} = -\frac{2}{3}.$$

**(f)** $\frac{d}{dx}\left[10\sin\left(\frac{\pi x}{2}\right)f^2(x)\right]$

$= 10\left(\sin\frac{\pi x}{2}\right)(2f(x)f'(x)) + 10f^2(x)\left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$

$= 20f(x)f'(x)\sin\frac{\pi x}{2} + 5\pi f^2(x)\cos\frac{\pi x}{2}$

At $x = 1$, the derivative is

$20f(1)f'(1)\sin\frac{\pi}{2} + 5\pi f^2(1)\cos\frac{\pi}{2}$

$= 20(-3)\left(\frac{1}{5}\right)(1) + 5\pi(-3)^2(0)$

$= -12.$

**67. (a)** $\frac{d}{dx}\left[3f(x) - g(x)\right] = 3f'(x) - g'(x)$

At $x = -1$, the derivative is

$3f'(-1) - g'(-1) = 3(2) - 1 = 5.$

**(b)** $\frac{d}{dx}\left[f^2(x)g^3(x)\right]$

$= f^2(x)\cdot 3g^2(x)g'(x) + g^3(x)\cdot 2f(x)f'(x)$

$= f(x)g^2(x)\left[3f(x)g'(x) + 2g(x)f'(x)\right]$

At $x = 0$, the derivative is

$f(0)g^2(0)\left[3f(0)g'(0) + 2g(0)f'(0)\right]$

$= (-1)(-3)^2\left[3(-1)(4) + 2(-3)(-2)\right]$

$= -9\left[-12 + 12\right] = 0.$

**(c)** $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$

At $x = -1$, the derivative is

$g'(f(-1))f'(-1) = g'(0)f'(-1) = (4)(2) = 8.$

**(d)** $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

At $x = -1$, the derivative is

$f'(g(-1))g'(-1) = f'(-1)g'(-1) = (2)(1) = 2.$

**(e)** $\frac{d}{dx}\frac{f(x)}{g(x)+2} = \frac{(g(x)+2)f'(x) - f(x)g'(x)}{(g(x)+2)^2}$

At $x = 0$, the derivative is

$\frac{(g(0)+2)f'(0) - f(0)g'(0)}{(g(0)+2)^2} = \frac{(-3+2)(-2) - (-1)(4)}{(-3+2)^2}$

$= 6.$

**(f)** $\frac{d}{dx}g(x+f(x)) = g'(x+f(x))\frac{d}{dx}(x+f(x))$

$= g'(x+f(x))(1+f'(x))$

At $x = 0$, the derivative is $g'(0+f(0))[1+f'(0)]$

$= g'(0-1)[1+(-2)] = (1)(-1) = -1$

**68.** $\frac{dw}{ds} = \frac{dw}{dr}\frac{dr}{ds} = \frac{d}{dr}[\sin(\sqrt{r}-2)]\frac{d}{ds}\left[8\sin\left(s+\frac{\pi}{6}\right)\right]$

$= \left[\cos(\sqrt{r}-2)\frac{1}{2\sqrt{r}}\right]\left[8\cos\left(s+\frac{\pi}{6}\right)\right]$

At $s = 0$, we have $r = 8\sin\left(0+\frac{\pi}{6}\right) = 4$ and so

$\frac{dw}{ds} = \left[\cos(\sqrt{4}-2)\frac{1}{2\sqrt{4}}\right]\left[8\cos\left(0+\frac{\pi}{6}\right)\right]$

$= \left(\frac{\cos 0}{4}\right)\left(8\cos\frac{\pi}{6}\right) = \left(\frac{1}{4}\right)\left(\frac{8\sqrt{3}}{2}\right) = \sqrt{3}$

**69.** Solving $\theta^2 t + \theta = 1$ for $t$, we have

$t = \frac{1-\theta}{\theta^2} = \theta^{-2} - \theta^{-1}$, and we may write:

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta}$$

$$\frac{d}{d\theta}(\theta^2+7)^{1/3} = \frac{dr}{dt}\frac{d}{d\theta}(\theta^{-2} - \theta^{-1})$$

$$\frac{1}{3}(\theta^2+7)^{-2/3}(2\theta) = \left(\frac{dr}{dt}\right)(-2\theta^{-3} + \theta^{-2})$$

$$\frac{dr}{dt} = \frac{2\theta(\theta^2+7)^{-2/3}}{3(-2\theta^{-3}+\theta^{-2})} = \frac{2\theta^4(\theta^2+7)^{-2/3}}{3(\theta-2)}$$

At $t = 0$, we may solve $\theta^2 t + \theta = 1$ to obtain $\theta = 1$,

and so $\frac{dr}{dt} = \frac{2(1)^4(1^2+7)^{-2/3}}{3(1-2)} = \frac{2(8)^{-2/3}}{-3} = -\frac{1}{6}.$

**70. (a)** One possible answer:

$x(t) = 10\cos\left(t+\frac{\pi}{4}\right),\ y(t) = 1$

**(b)** $s(0) = 10\cos\frac{\pi}{4} = 5\sqrt{2}$

**(c)** Farthest left:

When $\cos\left(t+\frac{\pi}{4}\right) = -1$, we have $s(t) = -10$.

Farthest right:

When $\cos\left(t+\frac{\pi}{4}\right) = 1$, we have $s(t) = 10$.

**70. Continued**

**(d)** Since $\cos\frac{\pi}{2}=0$, the particle first reaches the origin at $t=\frac{\pi}{4}$. The velocity is given by $v(t)=-10\sin\left(t+\frac{\pi}{4}\right)$, so the velocity at $t=\frac{\pi}{4}$ is $-10\sin\frac{\pi}{2}=-10$, and the speed at $t=\frac{\pi}{4}$ is $|-10|=10$. The acceleration is given by $a(t)=-10\cos\left(t+\frac{\pi}{4}\right)$, so the acceleration at $t=\frac{\pi}{4}$ is $-10\cos\frac{\pi}{2}=0$.

**71. (a)** $\frac{ds}{dt}=\frac{d}{dt}(64t-16t^2)=64-32t$

$\frac{d^2s}{dt^2}=\frac{d}{dt}(64-32t)=-32$

**(b)** The maximum height is reached when $\frac{ds}{dt}=0$, which occurs at $t=2$ sec.

**(c)** When $t=0,\ \frac{ds}{dt}=64$, so the velocity is 64 ft/sec.

**(d)** Since $\frac{ds}{dt}=\frac{d}{dt}(64t-2.6t^2)=64-5.2t$, the maximum height is reached at $t=\frac{64}{5.2}\approx 12.3\text{sec}$. The maximum height is $s\left(\frac{64}{5.2}\right)\approx 393.8$ ft.

**72. (a)** Solving $160=490t^2$, it takes $\frac{4}{7}$ sec. The average velocity is $\frac{160}{4}=280\,\text{cm/sec}$.

**(b)** Since $v(t)=\frac{ds}{dt}=980t$, the velocity is $(980)\left(\frac{4}{7}\right)=560$ cm/sec. Since $a(t)=\frac{dv}{dt}=980$, the acceleration is $980\ \text{cm}/\text{sec}^2$.

**73.** $\frac{dV}{dx}=\frac{d}{dx}\left[\pi\left(10-\frac{x}{3}\right)x^2\right]=\frac{d}{dx}\left[\pi\left(10x^2-\frac{1}{3}x^3\right)\right]$

$=\pi(20x-x^2)$

**74. (a)** $r(x)=\left(3-\frac{x}{40}\right)^2 x=9x-\frac{3}{20}x^2+\frac{1}{1600}x^3$

**(b)** The marginal revenue is

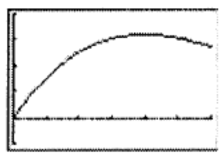
$r'(x)=9-\frac{3}{10}x+\frac{3}{1600}x^2$

$=\frac{3}{1600}(x^2-160x+4800)$

$=\frac{3}{1600}(x-40)(x-120)$,

which is zero when $x=40$ or $x=120$. Since the bus holds only 60 people, we require $0\le x\le 60$. The marginal revenue is 0 when there are 40 people, and the corresponding fare is $p(40)=\left(3-\frac{40}{40}\right)^2=\$4.00$.

**(c)** One possible answer:
If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y=r(x)$.



[0, 60] by [−50, 200]

**75. (a)** Since $x=\tan\theta$, we have $\frac{dx}{dt}=(\sec^2\theta)\frac{d\theta}{dt}=-0.6\sec^2\theta$. At point $A$, we have $\theta=0$ and $\frac{dx}{dt}=-0.6\sec^2 0=-0.6$ km/sec.

**(b)** $0.6\ \frac{\text{rad}}{\text{sec}}\cdot\frac{1\text{ revolution}}{2\pi\text{ rad}}\cdot\frac{60\text{sec}}{1\text{min}}=\frac{18}{\pi}$ revolutions per minute or approximately 5.73 revolutions per minute.

**76.** Let $f(x)=\sin(x-\sin x)$. Then

$f'(x)=\cos(x-\sin x)\frac{d}{dx}(x-\sin x)$

$=\cos(x-\sin x)(1-\cos x)$. This derivative is zero when $\cos(x-\sin x)=0$ (which we need not solve) or when $\cos x=1$, which occurs at $x=2k\pi$ for integers $k$. For each of these values, $f(x)=f(2k\pi)=\sin(2k\pi-\sin 2k\pi)=\sin(2k\pi-0)=0$. Thus, $f(x)=f'(x)=0$ for $x=2k\pi$, which means that the graph has a horizontal tangent at each of these values of $x$.

**77.** $y'(r)=\frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2l}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{r}\right)=-\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(l)=\frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2r}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{l}\right)=-\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(d)=\frac{d}{dd}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\right)\frac{d}{dd}(d^{-1/2})$

$=\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}d^{-3/2}\right)=-\frac{1}{4rl}\sqrt{\frac{T}{\pi d^3}}$

$y'(T)=\frac{d}{dT}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\right)\frac{d}{dt}(\sqrt{T})$

$=\frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\left(\frac{1}{2\sqrt{T}}\right)=\frac{1}{4rk\sqrt{\pi dT}}$

Since $y'(r)<0, y'(l)<0$, and $y'(d)<0$, increasing $r$, $l$, or $d$ would decrease the frequency. Since $y'(T)>0$, increasing $T$ would increase the frequency.

**78. (a)** $P(0)=\frac{200}{1+e^5}\approx 1$ student

**(b)** $\lim_{h\to\infty} P(t)=\lim_{h\to\infty}\frac{200}{1+e^{5-t}}=\frac{200}{1}=200$ students

**(c)** $P'(t)=\frac{d}{dt}200(1+e^{5-t})^{-1}$

$=1200(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$=\frac{200e^{5-t}}{(1+e^{5-t})^2}$

$$P''(t)=\frac{(1+e^{5-t})^2(200e^{5-t})(-1)-(200e^{5-t})(2)(1+e^{5-t})(e^{5-t})(-1)}{(1+e^{5-t})^4}$$

$=\frac{(1+e^{5-t})(-200e^{5-t})+400(e^{5-t})^2}{(1+e^{5-t})^3}$

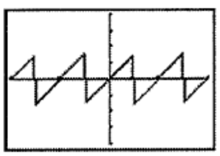
$=\frac{(200e^{5-t})(e^{5-t}-1)}{(1+e^{5-t})^3}$

Since $P''=0$ when $t=5$, the critical point of $y=P'(t)$ occurs at $t=5$. To confirm that this corresponds to the maximum value of $P'(t)$, note that $P''(t)>0$ for $t<5$ and $P''(t)<0$ for $t>5$. The maximum rate occurs at $t=5$, and this rate is

$P'(5)=\frac{200e^0}{(1+e^0)^2}+\frac{200}{2^2}=50$ students per day.

Note: This problem can also be solved graphically.

**79.**



$[-\pi, \pi]$ by $[-4, 4]$

**(a)** $x\neq k\frac{\pi}{4}$, where $k$ is an odd integer

**(b)** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

**(c)** Where it's not defined, at $x=k\frac{\pi}{4}$, $k$ an odd integer

**(d)** It has period $\frac{\pi}{2}$ and continues to repeat the pattern seen in this window.

**80.** Use implicit differentiation.

$x^2-y^2=1$

$\frac{d}{dx}(x^2)-\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$

$2x-2yy'=0$

$y'=\frac{2x}{2y}=\frac{x}{y}$

$y''=\frac{d}{dx}\frac{x}{y}$

$=\frac{(y)(1)-(x)(y')}{y^2}$

$=\frac{y-x\left(\frac{x}{y}\right)}{y^2}$

$=\frac{y^2-x^2}{y^3}$

$=-\frac{1}{y^3}$

(since the given equation is $x^2-y^2=1$)

At $(2, \sqrt{3})$, $\frac{d^2y}{dx^2}=-\frac{1}{y^3}=-\frac{1}{(\sqrt{3})^3}=-\frac{1}{3\sqrt{3}}$.

**81. (a)** $v(t)=\frac{ds}{dt}=\frac{d}{dt}(t^3-2t+3)$

$v(t)=3t^2-2$

**(b)** $a(t)=\frac{dv}{dt}=\frac{d}{dt}=(3t^2-2)$

$a(t)=6t$

**(c)** $v(t)=3t^2-2=0$

$t^2=\frac{2}{3}$

$t=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$

$t^2=\frac{2}{3}$

$t=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$

**(d)** $v(t)=3t^2-2<0$

$3t^2<2$

$t<\frac{\sqrt{6}}{3}$, and $t>0$

$0<t<\frac{\sqrt{6}}{3}$

**81. Continued**

**(e)** $v(t) = 3t^2 - 2 > 0$

$3t^2 > 2$

$t > \dfrac{\sqrt{6}}{2}$

**82. (a)** $\dfrac{d}{dx}e^u = e^u \dfrac{du}{dx}$ where $u = x$

$\dfrac{d}{dx}\dfrac{e^x + e^{-x}}{2} = \dfrac{e^x - e^{-x}}{2}$

**(b)** $\dfrac{d}{dx}\dfrac{e^x - e^{-x}}{2} = \dfrac{e^x + e^{-x}}{2}$

**(c)** $y(1) = \dfrac{e^1 + e^{-1}}{2} = 1.543$

$y'(1) = \dfrac{e^1 - e^{-1}}{2} = 1.175$

$y = 1.175(x-1) + 1.543$

$y = 1.175x + 0.368$

**(d)** $m_2 = -\dfrac{1}{m_1} = -\dfrac{1}{1.175} = -0.851$

$y = -0.851(x-1) + 1.543$

$y = -0.851x + 2.394$

**(e)** $y' = 0 = \dfrac{e^x - e^{-x}}{2}$

$0 = e^x - e^{-x}$

$e^x = e^{-x}$

$x = -x$ or $x = 0$

**83. (a)** $1 - x^2 > 0$

$x^2 > 1, -1 < x < 1$

**(b)** $f'(x) = \dfrac{d}{dx}\ln(1-x^2)$ $\qquad u = 1 - x^2$

$\dfrac{d}{dx}\ln(u) = \dfrac{1}{u}\dfrac{du}{dx}$ $\qquad \dfrac{du}{dx} = -2x$

$= \dfrac{-2x}{(1-x^2)}$

**(c)** $1 - x^2 > 0, \ -1 < x < 1$

**(d)** $y'\left(\dfrac{1}{2}\right) = \dfrac{-2\left(\dfrac{1}{2}\right)}{1-\left(\dfrac{1}{2}\right)^2} = -\dfrac{1}{3/4} = -4/3$

# Chapter 4

## Applications of Derivatives

### **Section 4.1** Extreme Values of Functions (pp. 187–195)

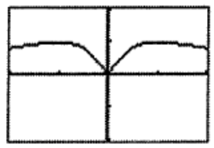
#### Exploration 1 Finding Extreme Values

**1.** From the graph we can see that there are three critical points: $x = -1, 0, 1$.

Critical point values: $f(-1) = 0.5$, $f(0) = 0$, $f(1) = 0.5$

Endpoint values: $f(-2) = 0.4$, $f(2) = 0.4$

Thus $f$ has absolute maximum value of 0.5 at $x = -1$ and $x = 1$, absolute minimum value of 0 at $x = 0$, and local minimum value of 0.4 at $x = -2$ and $x = 2$.



$[-2, 2]$ by $[-1, 1]$

**2.** The graph of $f'$ has zeros at $x = -1$ and $x = 1$ where the graph of $f$ has local extreme values. The graph of $f'$ is not defined at $x = 0$, another extreme value of the graph of $f$.

$[-2, 2]$ by $[-1, 1]$

**3.** Using the chain rule and $\dfrac{d}{dx}(|x|) = \dfrac{|x|}{x}$, we find

$\dfrac{df}{dx} = \dfrac{|x|}{x} \cdot \dfrac{1-x^2}{(x^2+1)^2}$.

#### Quick Review 4.1

**1.** $f'(x) = \dfrac{1}{2\sqrt{4-x}} \cdot \dfrac{d}{dx}(4-x) = \dfrac{-1}{2\sqrt{4-x}}$

**2.** $f'(x) = \dfrac{d}{dx}2(9-x^2)^{-1/2} = -(9-x^2)^{-3/2} \cdot \dfrac{d}{dx}(9-x^2)$

$= -(9-x^2)^{-3/2}(-2x) = \dfrac{2x}{(9-x^2)^{3/2}}$

**3.** $g'(x) = -\sin(\ln x) \cdot \dfrac{d}{dx}\ln x = -\dfrac{\sin(\ln x)}{x}$

**4.** $h'(x) = e^{2x} \cdot \dfrac{d}{dx}2x = 2e^{2x}$

**5.** Graph (c), since this is the only graph that has positive slope at $c$.

**6.** Graph (b), since this is the only graph that represents a differentiable function at $a$ and $b$ and has negative slope at c.

**7.** Graph (d), since this is the only graph representing a function that is differentiable at $b$ but not at $a$.

**8.** Graph (a), since this is the only graph that represents a function that is not differentiable at $a$ or $b$.

**9.** As $x \to 3^-$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x \to 3^-} f(x) = \infty$.

**10.** As $x \to 3^+$, $\sqrt{9-x^2} \to 0^+$. Therefore, $\lim_{x \to -3^+} f(x) = \infty$.